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Abstract

We introduce a generalization of Glimm's random choice method, which provides us with an approximation of entropy solutions to quasilinear hyperbolic system of balance laws. The flux-function and the source term of the equations may depend on the unknown as well as on the time and space variables. The method is based on local approximate solutions of the generalized Riemann problem, which form building blocks in our scheme and allow us to take into account naturally the effects of the flux and source terms. To establish the nonlinear stability of these approximations, we investigate nonlinear interactions between generalized wave patterns. This analysis leads us to a global existence result for quasilinear hyperbolic systems with source-term, and applies, for instance, to the compressible Euler equations in general geometries and to hyperbolic systems posed on a Lorentzian manifold.

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1. Introduction

1.1 Hyperbolic systems of balance laws. This paper[3] is concerned with the approximation of entropy solutions to the Cauchy problem for a quasilinear hyperbolic system

$$\partial_t u + \partial_x f(t,x,u) = g(t,x,u), \quad t > 0,\ x \in \mathbb{R}, \tag{1.1}$$

$$u(0,x) = u_0(x), \quad x \in \mathbb{R}, \tag{1.2}$$

where $u = u(t,x) \in \mathbb{R}^p$ is the unknown. We propose here a generalized version of the Glimm scheme [10] which allows us to deal with a large class of mappings f, g and take into account the geometric effect of the flux and source terms. Our scheme is based on an approximate solver for the *generalized Riemann problem*, based on an asymptotic expansion introduced by LeFloch and Raviart [17]. The approach provides high accuracy and stability, under mild restrictions on the equation and the data.

In (1.1), the flux $f = f(t,x,u) \in \mathbb{R}^p$ and the source-term $g = g(t,x,u) \in \mathbb{R}^p$ are given smooth maps defined for all $(t,x,u) \in \mathbb{R}_+ \times \mathbb{R} \times \mathcal{U}$, where $\mathcal{U}$ is a small neighborhood of the origin in $\mathbb{R}^p$, and the initial data $u_0 : \mathbb{R} \to \mathcal{U}$ is a function with bounded total variation. We assume that the Jacobian matrix $A(t,x,u) := \frac{Df}{Du}(t,x,u)$ admits p real and distinct eigenvalues,

$$\lambda_1(t,x,u) < \lambda_2(t,x,u) < \ldots < \lambda_p(t,x,u),$$

and therefore a basis of right-eigenvectors $r_j(t,x,u)$ $(1 \leq j \leq p)$, Finally, we assume that each characteristic field is either genuinely nonlinear $(\nabla\lambda_j(t,x,u) \cdot r_i(t,x,u) \neq 0)$ or linearly degenerate $(\nabla\lambda_j(t,x,u) \cdot r_i(t,x,u) = 0)$.

One important motivation for considering general balance laws (1.1) comes from the theory of general relativity. In this context, the vector u typically consists of fluid variables as well as (first order derivatives) of the coefficients of an unknown, Lorentzian metric tensor. (See [3, 5] and the reference therein.) One can also freeze the metric coefficients and concentrate on the dynamics of the fluid. For instance, the compressible Euler equations describing the dynamics of a gas flow in general geometry read:

$$\begin{aligned}
&\partial_t \rho + \partial_x(\rho v) = -\frac{\partial_x a}{a}\rho v - \frac{\partial_t a}{a}\rho, \\
&\partial_t(\rho v) + \partial_x(\rho v^2 + p) = -\frac{\partial_x a}{a}(\rho v^2) - \frac{\partial_t a}{a}\,\rho v, \\
&\partial_t(\rho E) + \partial_x(\rho v E + p v) = -\frac{\partial_x a}{a}(\rho v E + p v) - \frac{\partial_t a}{a}\,\rho E
\end{aligned} \tag{1.3}$$

[3]This paper is based on notes written by the second author in July 1990, for his Habilitation memoir (Chapter XI) at the University of Paris VI.

where $a = a(t,x) > 0$ can be regarded as the cross section of a time-dependent (moving) duct, and ρ, v, $p(\rho, e)$, e, and $E = e + u^2/2$ are the density, velocity, pressure, internal energy, and total energy of the gas, respectively. The system (1.3) describes a situation where the fluid does not affect the variation of the duct; i.e. the function $a(t,x)$ is given and, for simplicity, smooth. The system (1.3) is of the form (1.1) with $u = (\rho, \rho v, \rho E)^T$, $f = f(u) = (\rho v, \rho v^2 + p, \rho v E + pv)^T$ and $g = g(t,x,u) = -\frac{\partial_x a}{a} g_1(u) - \frac{\partial_t a}{a} u$ where $g_1(u) = (\rho v, \rho v^2, \rho v E + pv)^T$.

We are interested in solutions to (1.1)-(1.2) which have bounded total variation in space for all times and satisfy the equations in the sense of distributions, together with an entropy condition [15, 8, 16]. In the special case that

$$f = f(u), \quad g = 0,$$

the existence of global entropy solutions was established by Glimm [10], assuming that the initial data $u_0(x)$ has sufficiently small total variation. Recall that two main ingredients in Glimm's random choice method are (1) the solutions of Riemann problems and (2) a projection step based on a sequence of randomly chosen points.

Let us first indicate some of the earlier work on the subject. The system (1.1) with

$$f = f(x,u), \quad g = g(x,u),$$

was treated in pioneering work by Liu [20, 21], via a suitable extension of the Glimm method: the approximate solutions are defined by pasting together steady state solutions, i.e., solutions $v = v(x)$ of the ordinary differential equation

$$\frac{d}{dx}\big(f(x,v)\big) = g(x,v).$$

He established the existence of solutions defined in a finite interval of time $[0,T)$ as long as either T or the L^1 norms of g and $\partial g/\partial u$ are sufficiently small. Next, assuming in addition that the eigenvalues of the matrix $A(x,u)$ never vanish (so that no resonance takes place), Liu deduced a global existence result (with $T = +\infty$). Steady-state solutions were also used in the work by Glimm, Marshall, and Plohr [12].

For more general mappings f, g, the existence for (1.1)-(1.2) is established by Dafermos and Hsiao [7] and Dafermos [8, 9]. They assume that $f_x(u^*, t, x) = g(u^*, t, x) = 0$ at some (equilibrium) constant state u^*, hence u^* is a solution of (1.1) around which (1.1) can be formally linearized. They also require that the linearized system satisfies a dissipative property. Their main result concerns the consistency and stability of a generalization of the Glimm method, yielding therefore the global existence of entropy solutions to (1.1). In [7], the approximate solutions to the Cauchy problem on each time step are based on classical Riemann solutions with initial data suitably modified by both the source term g and the map $\theta := A^{-1} f_x$.

Next, Amadori et al. [1, 2] developed further techniques to establish the existence of solutions for a large class of systems having $f = f(u)$ and $g = g(x, u)$, and discussed Dafermos-Hsiao dissipative condition. For some particular systems (of two or three equations) the condition that the total variation be small can be relaxed; see for instance Luskin and Temple [22], Groah and Temple [11], Barnes, LeFloch, Schmidt, and Stewart [3], and the references cited therein. In these papers, the decreasing of a total variation functional (measured with respect to a suitable chosen coordinate) was the key to establish the stability of the scheme.

1.2 A new version of the Glimm method. In the present paper we provide an alternative approach to Dafermos-Hsiao's method, and introduce a generalized version of the Glimm scheme for general mappings f, g. Integrability assumptions will be required (and discussed later on) on the matrix A and the mapping $q : \mathbb{R}_+ \times \mathbb{R} \times \mathcal{U} \to \mathbb{R}^p$ defined by

$$q(t, x, u) := g(t, x, u) - \frac{\partial f}{\partial x}(t, x, u). \tag{1.4}$$

It should be emphasized that only this combination of the source and the flux will be important in our approach, which can be summarized as follows.

First, we study the generalized Riemann problem associated with the system (1.1), i.e. the Cauchy problem with piecewise constant initial data. The existence of solutions defined locally in spacetime in a neighborhood of the initial discontinuity was studied in Li and Yu [18] and Harabetian [13]. Contrary to the case where f, g only depend upon the unknown u, no closed formula is available for the solutions of the generalized Riemann problem. We propose here an *approximate Riemann solver*, inspired by a technique of asymptotic expansion introduced by Ben-Artzi and Falcovitz [4] (for the gas dynamics equations) and LeFloch and Raviart [17] (for general hyperbolic systems of balance laws); see also [6].

Our scheme for solving approximately the generalized Riemann problem can be re-interpreted as a splitting algorithm (the hyperbolic operator and the source term being decoupled). Since an approximate (rather than an exact) solution to the generalized Riemann problem is used, it is crucial to establish an error estimate which we achieve in Proposition 2.1 below, under a mild assumption on the data u_0, f, g. This estimate will be necessary to ensure the consistency of our generalized Glimm method.

Second, we study the nonlinear interaction of waves issuing from two generalized Riemann problems, and establish a suitable extension of Glimm's estimates [10] to the general system (1.1); cf. Proposition 3.3. This is a key, technical part of our analysis.

Third, we introduce our scheme and prove its stability in total variation, under the assumption that the initial data u_0 has sufficiently small total variation and that the total amplification due to (the derivatives of) f, g to the total variation of the solution is sufficiently small; cf. Theorem 4.3. More precisely, we impose that

$$\frac{\partial^2 A}{\partial t \partial u}, \quad \frac{\partial^2 A}{\partial x \partial u}, \quad q, \quad \frac{\partial q}{\partial u}$$

are sufficiently small in $L^1(\mathbb{R}_+ \times \mathbb{R})$.

Finally, we conclude with the convergence of the proposed scheme (Cf. Theorem 5.1) which yields the global existence of entropy solutions for the Cauchy problem (1.1)-(1.2). The solution satisfies an entropy inequality and has bounded total variation in x for all $t \geq 0$. Our results cover in particular the case

$$\partial_t u + \partial_x f(u) = g(t), \tag{1.5}$$

for which global existence of entropy solutions is established under the sole assumption

$$\int_0^{+\infty} |g(t)| \, dt << 1. \tag{1.6}$$

Without further restriction on the flux f, this condition is clearly necessary in order to apply the Glimm method, since, for instance in the trivial case $p = 1$ and $f = 0$, (1.5) reduces to the differential equation

$$\partial_t u = g(t). \tag{1.7}$$

On one hand, the condition (1.6) holds if and only if every solution of (1.7) remains close to a constant state, which is a necessary condition in order to apply the Glimm method. On the other hand, when one of the eigenvalues of the system (1.1) vanishes, the amplitude of solutions could become arbitrarily large and the solutions would not remain bounded —except when the source term satisfies a "damping" property in time.

As a direct application, the global existence of entropy solutions to (1.3) follows, if the source g and its derivative $\frac{\partial g}{\partial u}$ are sufficiently small in $L^1(\mathbb{R}_+ \times \mathbb{R})$, which is the case, for instance, if the support of (a_t, a_x) is sufficiently small.

2. An approximate solver for the generalized Riemann problem

In the present section, we introduce an approximate solution to the generalized Riemann problem associated with the system (1.1), and we derive an error estimates (see Proposition 2.1 below).

Given $t_0 > 0$, $x_0 \in \mathbb{R}$, and two constant states $u_L, u_R \in \mathbb{R}^p$, we consider the *generalized Riemann problem*, denoted by $R_G(u_L, u_R; t_0, x_0)$, and consisting of the following equations and initial conditions:

$$\partial_t u + \partial_x f(t, x, u) = g(t, x, u), \quad t > t_0, \ x \in \mathbb{R}, \tag{2.1}$$

$$u(0,x) = \begin{cases} u_L, & x < x_0, \\ u_R, & x > x_0. \end{cases} \tag{2.2}$$

Replacing f and g in (2.1) by $f(t_0, x_0, u)$ and 0, respectively, the problem $R_G(u_L, u_R; t_0, x_0)$ reduces to the *classical Riemann problem*,which we denote by $R_C(u_L, u_R; t_0, x_0)$, that is the equations

$$\partial_t u + \partial_x f(t_0, x_0, u) = 0, \quad u(t,x) \in \mathbb{R}^p, \quad t > t_0, \quad x \in \mathbb{R} \tag{2.3}$$

together with the initial data (2.2). This problem was solved by Lax under the assumption that the initial jump $|u_R - u_L|$ be sufficiently small: the solution to $R_C(u_L, u_R; t_0, x_0)$ is self-similar (i.e. depends only on $\frac{x-x_0}{t-t_0}$) and consists of at most $(p+1)$ constant states $u_L = u_0, u_1, \ldots, u_p = u_R$, separated by rarefaction waves, shock waves or contact discontinuities; see Figure 2.1.

The following terminology and notation will be used throughout this paper. Let $W_C = W_C(\xi; u_L, u_R; t_0, x_0)$ be the solution of $R_C(u_L, u_R; t_0, x_0)$ with $\xi = (x - x_0)/(t - t_0)$. We say that the problem $R_C(u_L, u_R; t_0, x_0)$ is solved by the elementary waves (u_{i-1}, u_i) $(i = 1, \ldots, p)$ if each constant state u_i belongs to the *i-wave curve* $\mathcal{W}_i(u_{i-1})$ issued from the state u_{i-1} in the phase space, and (u_{i-1}, u_i) is called an $i-wave$ of $R_C(u_L, u_R; t_0, x_0)$. When the i-characteristic field is genuinely nonlinear, the curve $\mathcal{W}_i(u_{i-1})$ consists of two parts, the i-rarefaction curve and the i-shock curve issuing from u_{i-1}; if i-characteristic field is linearly degenerate, the curve $\mathcal{W}_i(u_{i-1})$ is a C^2 curve of i-contact discontinuities. Call ε_i the strength of the i-wave (u_{i-1}, u_i) along the i-curve, so that, for a genuinely nonlinear i-field, we can assume that $\varepsilon_i \geq 0$ if (u_{i-1}, u_i) is a rarefaction wave, and $\varepsilon_i \leq 0$ if (u_{i-1}, u_i) is a shock wave. On the other hand, ε_i has no specific sign if (u_{i-1}, u_i) is a contact discontinuity.

Let $\varepsilon_i(u_L, u_R; t_0, x_0)$ denote the *wave strength* of the i-wave (u_{i-1}, u_i) in the Riemann problem $R_C(u_L, u_R; t_0, x_0)$, and vector $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_p)$ denote the *wave strength* of $R_C(u_L, u_R; t_0, x_0)$ (so $|\varepsilon|$ is equivalent to the total variation of

$$W_C(\xi; u_L, u_R; t_0, x_0)).$$

In addition, we let $\sigma_i^- = \lambda_i(u_{i-1}, t_0, x_0)$ and $\sigma_i^+ = \lambda_i(u_i, t_0, x_0)$ be the lower and upper speeds of the i-rarefaction wave (u_{i-1}, u_i) respectively, and σ_i be the speed of the i-shock or i-contact discontinuity. If the i-wave is a shock or a contact discontinuity we set $\sigma_i^- = \sigma_i^+ = \sigma_i$.

From the implicit function theorem we deduce that the states u_i and the speeds $\sigma_i^\pm$ are smooth functions of u_L, u_R, t_0, and x_0. Moreover, one can check that $u_i = u_L + O(1)|u_R - u_L|$ $(i = 0, 1, \ldots, p)$, and, for an i-shock (u_{i-1}, u_i),

$$\sigma_i = \lambda_i(u_{i-1}; t_0, x_0) + O(1)\,|u_i - u_{i-1}|, \quad i = 1, 2, \ldots, p,$$

where $O(1)$ is bounded function possibly depending on $u_L, u_R \in \mathcal{U}$, $t_0 \geq 0$, and $x_0 \in \mathbb{R}$.

Consider next the generalized Riemann problem on which a large literature is available [18, 13, 4, 6, 17]. First, we recall [18] that the solution of $R_G(u_L, u_R; t_0, x_0)$ is piecewise smooth and has a local structure which is similar to the one of the associated classical Riemann problem $R_C(u_L, u_R; t_0, x_0)$. Following [17] we consider an *approximate Riemann solution* of the problem $R_G(u_L, u_R; t_0, x_0)$, denoted by $W_G(t, x; u_L, u_R; t_0, x_0)$ and defined by

$$W_G(t, x; u_L, u_R; t_0, x_0) = W_C(\xi) + (t - t_0)\, q(t_0, x_0, W_C(\xi)) \tag{2.4}$$

for $t > t_0$ and $x \in \mathbb{R}$. Here, the function $q(t, x, u)$ is given by (1.4), and

$$\xi = \frac{x - x_0}{t - t_0}, \quad W_C(\xi) = W_C(\xi; u_L, u_R; t_0, x_0).$$

Observe that the function $W_G(t, x; u_L, u_R; t_0, x_0)$ is constructed as a superposition of the corresponding classical Riemann solution $W_C(\xi; u_L, u_R; t_0, x_0)$ and an asymptotic expansion term $(t - t_0)q(t_0, x_0, W_C(\xi))$ (see Figure 2.2).

Within a region where function $W_C(\xi)$ is a constant, the function

$$W_G(t, x; u_L, u_R; t_0, x_0)$$

is a linear function of t, namely,

$$W_G(t, x; u_L, u_R; t_0, x_0) = u_i + (t - t_0)q(t_0, x_0, u_i), \quad \sigma_i^+ < \frac{x}{t} < \sigma_{i+1}^- \tag{2.5}$$

for $i = 0, 1, \ldots, p$. By convention, $\sigma_0^+ := -\infty$ and $\sigma_{p+1}^- := +\infty$. Whenever there will be no ambiguity, we will use the notation $W_G(t, x)$ or $W_G(t, x; u_L, u_R)$ for $W_G(t, x; u_L, u_R; t_0, x_0)$.

To describe the structure of $W_G(t, x; u_L, u_R; t_0, x_0)$, it is convenient to say that the approximate solution $W_G(t, x, u_L, u_R; t_0, x_0)$ consists of an i-wave (u_{i-1}, u_i) if (u_{i-1}, u_i) is an i-wave of the corresponding classical Riemann solution

$$W_C(\xi; u_L, u_R; t_0, x_0).$$

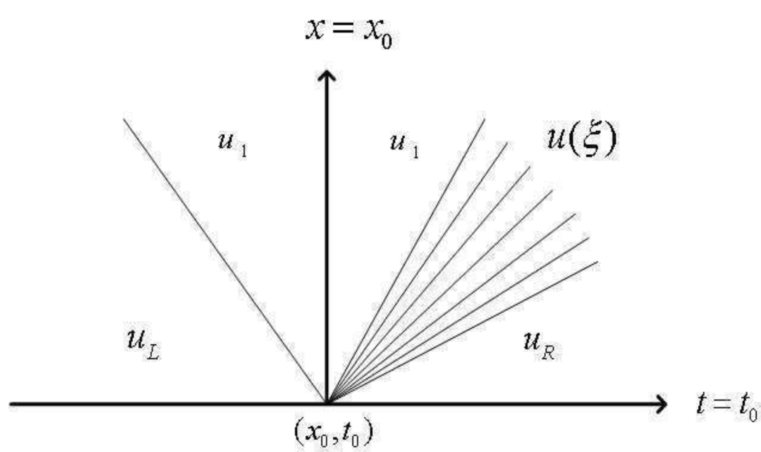

Figure 2.1 : Classical Riemann solution (p=2),
u_L, u_1 and u_R are constant states, $u = u(\xi)$ is a function of $\xi = \frac{x}{t}$.

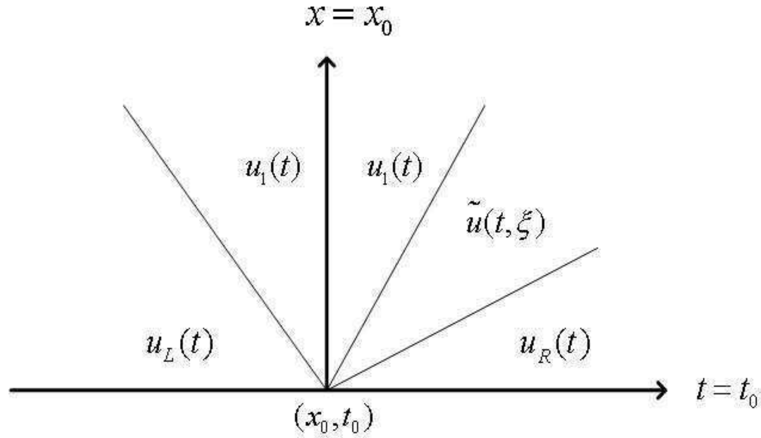

Figure 2.2 : Generalized Riemann solution (p=2),
$u_L(t)$, $u_1(t)$, $u_R(t)$ are functions of t and $\tilde{u}(t,\xi)$ is constructed by (2.4).

We now prove that the function $W_G(t,x)$ defined in (2.4) approximately solves the problem $R_G(u_L, u_R; t_0, x_0)$, by evaluating the discrepancy between $W_G(t,x)$ and the exact solution of $R_G(u_L, u_R; t_0, x_0)$. Given any $s > 0$ and $r > 0$, and any C^1 function $\theta : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ with compact support, we now show that the term

$$\Delta(s,r;\theta) := \int_{t_0}^{t_0+s} \int_{x_0-r}^{x_0+r} \{W_G\, \partial_t\theta + f(t,x,W_G)\, \partial_x\theta + g(t,x,W_G)\, \theta\Big)\, dxdt \quad (2.6)$$

is of third order in r, s, provided that the condition (2.7) holds.

Proposition 2.1. *Let $\theta : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ be a compactly supported, C^1 function. Then, for every $(t_0, x_0) \in \mathbb{R}_+ \times \mathbb{R}$, $u_L, u_R \in \mathcal{U}$, and any positive numbers s, r satisfying the (Courant-Friedrichs-Levy -type) stability condition*

$$\frac{s}{r} \sup |\lambda_i(t,x,u)| \le 1 \quad (2.7)$$

(the supremum being taken over $1 \le i \le p$, $(t,x) \in \mathbb{R}_+ \times \mathbb{R}$, and $u \in \mathcal{U}$), the function $W_G(t,x) = W_G(t,x; u_L, u_R; t_0, x_0)$ satisfies

$$\begin{aligned}
\Delta(s,r;\theta) \;=\; & \int_{x_0-r}^{x_0+r} W_G(t_0+s,\cdot)\theta(t_0+s,\cdot)dx - \int_{x_0-r}^{x_0+r} W_G(t_0,\cdot)\theta(t_0,\cdot)dx \\
& + \int_{t_0}^{t_0+s} f(\cdot, x_0+r, W_G(\cdot, x_0+r))\theta(\cdot, x_0+r)dt \\
& - \int_{t_0}^{t_0+s} f(\cdot, x_0-r, W_G(\cdot, x_0-r))\theta(\cdot, x_0-r)dt \\
& + O(1)(s^2+r^2)(s+r+|u_R-u_L|)||\theta||_{C^1},
\end{aligned} \quad (2.8)$$

where $\Delta(s, r; \theta)$ *is given in* (2.6) *and* $||\theta||_{\mathcal{C}^1} = ||\theta||_{\mathcal{C}^0} + ||\partial_t\theta||_{\mathcal{C}^0} + ||\partial_x\theta||_{\mathcal{C}^0}$.

The left-hand side of (2.8) vanishes when $W_G(t, x)$ is replaced by the exact solution of $R_G(u_L, u_R; t_0, x_0)$. Thus, the right hand side of (2.8) represents the error due to the choice of approximate solution $W_G(t, x)$.

Remark 2.2. 1. Condition (2.7) ensures that the waves in $R_C(u_L, u_R; t_0, x_0)$ can not reach the lines $\{x = x_0 \pm r\}$ for $t \leq t_0 + s$, so that the waves in the rectangle region $D_{(t_o,x_0)} \equiv [x_0 - r, x_0 + r] \times [t_0, t_0 + s)$ do not interact with the waves outside $D_{(t_o,x_0)}$.

2. In a different context, Liu [20] derived earlier an estimate similar to (2.7), but for an approximation based on steady state solutions of the hyperbolic system and with initial data consisting of two steady state solutions of (2.1) (with $f = f(u)$ and $g = g(x, u)$).

3. Our formula (2.11) yields a possible generalization to the class of quasilinear systems (1.1) of the notion of (classical) Riemann solver introduced by Harten and Lax in [14].

4. One can check similarly that W_G satisfies an entropy inequality associated with an entropy pair (when available). The error terms are completely similar to those found in (2.11). This will be used to show that the weak solution generated by the random choice method satisfies all the entropy inequalities.

Proof. Without loss of generality, we can assume that $(t_0, x_0) = (0, 0)$. Given a C^1 function θ with compact support in $\mathbb{R}_+ \times \mathbb{R}$, we define $m(t, x) := W_G\partial_t\theta + f(t, x, W_G)\partial_x\theta + g(t, x, W_G)\theta$. From (2.6) we have $\Delta(s, r; \theta) = \int_0^s \int_{-r}^r m(t, x)dxdt$. Next, we decompose $\Delta(s, r; \theta)$ as

$$\Delta(s, r; \theta) = \sum_{i=0}^{p} \Delta_i^1(s, r; \theta) + \sum_{\substack{i-rare. \\ waves}} \Delta_i^2(s, r; \theta) \tag{2.9}$$

where

$$\Delta_i^1(s, r; \theta) := \int_0^s \int_{\sigma_i^+ t}^{\sigma_{i+1}^- t} m(t, x)\, dxdt, \quad 1 \leq i \leq p - 1,$$

$$\Delta_0^1(s, r; \theta) := \int_0^s \int_{-r}^{\sigma_1^- t} m(t, x)\, dxdt, \qquad \Delta_p^1(s, r; \theta) := \int_0^s \int_{\sigma_p^+ t}^{r} m(t, x)\, dxdt,$$

and (if the i-wave, $1 \leq i \leq p$, is a rarefaction wave)

$$\Delta_i^2(s, r; \theta) := \int_0^s \int_{\sigma_i^- t}^{\sigma_i^+ t} m(t, x)\, dxdt$$

We first compute Δ_i^1 in the region where classical Riemann solution W_C is a constant state. According to the form of $W_G(t,x)$ in (2.5), it follows that

$$W_G(t,x) = u_i + t\, q(0,0;u_i) \tag{2.10}$$

for $\frac{x}{t} \in [\sigma_i^+, \sigma_{i+1}^-]$, $i \in \{1, 2, \dots, p-1\}$. By a simple calculation and the definition of q in (1.4), we have

$$\partial_t W_G + \partial_x f(t,x,W_G) - g(t,x,W_G) = q(0,0;u_i) - q(t,x,W_G)$$

for $i \in \{1, 2, \dots, p-1\}$. By multiplication by the function θ and then using integration by parts, we obtain

$$\begin{aligned} \Delta_i^1(s,r;\theta) \ =\ & \int_{\sigma_i^+ s}^{\sigma_{i+1}^- s} W_G(s,x)\theta(s,x)dx \\ & + \int_0^s (f(t,\sigma_{i+1}^- t, W_G(t,\sigma_{i+1}^- t)) - \sigma_{i+1}^- W_G(t,\sigma_{i+1}^- t))\theta(t,\sigma_{i+1}^- t)dt \\ & - \int_0^s (f(t,\sigma_i^+ t, W_G(t,\sigma_i^+ t)) - \sigma_i^+ W_G(t,\sigma_i^+ t))\theta(t,\sigma_i^+ t)dt \\ & - \int_0^s \int_{\sigma_i^+ t}^{\sigma_{i+1}^- t} \big(q(0,0;u_i) - q(t,x,W_G)\big)\,\theta(t,x)\,dxdt. \end{aligned} \tag{2.11}$$

By the property that q is Lipschitz continuous with respect to t, x and u on the compact set $[0,s] \times [-r,r]$ and the form of $W_G(t,x)$ in (2.10), the last term on the right hand side of (2.11) can be estimated by $O(s^3)\,||\theta||_{\mathcal{C}^0}$ with the bound $O(1)$ depending on q. Therefore, equality (2.11) leads to

$$\begin{aligned} \Delta_i^1(s,r;\theta) \ =\ & \int_{\sigma_i^+ s}^{\sigma_{i+1}^- s} W_G(s,x)\theta(s,x)dx \\ & + \int_0^s (f(t,\sigma_{i+1}^- t, W_G(t,\sigma_{i+1}^- t)) - \sigma_{i+1}^- W_G(t,\sigma_{i+1}^- t-))\theta(t,\sigma_{i+1}^- t)dt \\ & - \int_0^s (f(t,\sigma_i^+ t, W_G(t,\sigma_i^+ t+0)) - \sigma_i^+ W_G(t,\sigma_i^+ t+))\theta(t,\sigma_i^+ t)dt \\ & + O(1)s^3\,||\theta||_{\mathcal{C}^0} \end{aligned} \tag{2.12}$$

for $i = 1, 2, \dots, p-1$. In the same fashion one can show that

$$\begin{aligned} \Delta_0^1(s,r;\theta) \ =\ & \int_{-r}^{\sigma_1^- s} W_G(s,x)\theta(s,x)dx - \int_{-r}^0 W_G(0,x)\theta(0,x)dx \\ & + \int_0^s (f(t,\sigma_1^- t, W_G(t,\sigma_1^- t-)) - \sigma_1^- W_G(t,\sigma_1^- t-))\theta(t,\sigma_1^- t)dt \\ & - \int_0^s f(t,-r,W_G(t,-r))\theta(t,-r)dt \\ & + O(1)s^2(s+r)||\theta||_{\mathcal{C}^0} + O(1)sr^2||\theta||_{\mathcal{C}^0}, \end{aligned} \tag{2.13}$$

and

$$
\begin{aligned}
\Delta_p^1(s,r;\theta) \;=\; & \int_{\sigma_p^+ s}^{r} W_G(s,x)\theta(s,x)dx - \int_0^r W_G(0,x)\theta(0,x)dx \\
& + \int_0^s f(t,r,W_G(t,r))\theta(t,r)dt \\
& - \int_0^s (f(t,\sigma_p^+ t, W_G(t,\sigma_p^+ t+)) - \sigma_p^+ W_G(t,\sigma_p^+ t+))\theta(t,\sigma_p^+ t)dt \\
& + O(1)s^2(s+r)||\theta||_{\mathcal{C}^0} + O(1)sr^2||\theta||_{\mathcal{C}^0}.
\end{aligned} \tag{2.14}
$$

Next, suppose that $W_C(t,x)$ consists of an i-rarefaction wave in the region $\big\{(t,x)|\frac{x}{t}\in[\sigma_i^-,\sigma_i^+]\big\}$ for some $i\in 1,\dots,p$. It follows that $W_G(t,x)$ in this region is of the form

$$
W_G(t,x) = \widetilde{W}_C(\frac{x}{t}) + t\,q(0,0;\widetilde{W}_C(\frac{x}{t}))
$$

where $\widetilde{W}_C(\frac{x}{t})$ is the i-rarefaction wave of the classical Riemann problem

$$
R_C(u_L,u_R;t_0,x_0).
$$

By setting $\xi=\frac{x}{t}$, $W_G(t,x)=\widetilde{W}_G(t,\xi)$, and the technique of change of variables $(t,x)\to(t,\xi)$, we obtain

$$
\begin{aligned}
& \partial_t W_G + \partial_x f(t,x,W_G) - g(t,x,W_G) \\
& = \partial_t\widetilde{W}_G - \frac{\xi}{t}\partial_\xi\widetilde{W}_G + \frac{1}{t}\partial_\xi f(t,t\xi,\widetilde{W}_G) - g(t,t\xi,\widetilde{W}_G) \\
& = \frac{1}{t}\Big(\frac{\partial f}{\partial u}(t,t\xi,\widetilde{W}_G) - \xi I)(I + t\frac{\partial q}{\partial u}(0,0,\widetilde{W}_C)\Big)\cdot\frac{d\widetilde{W}_C}{d\xi} + q(0,0,\widetilde{W}_C) - q(t,t\xi,\widetilde{W}_C)
\end{aligned} \tag{2.15}
$$

where I is the $p\times p$ identity matrix. Since $\widetilde{W}_C(\xi)$ is a rarefaction wave for the system (2.3), this implies that

$$
\frac{1}{t}\Big(-\xi\cdot I + \frac{\partial f}{\partial u}(0,0,\widetilde{W}_C)\Big)\cdot\frac{d\widetilde{W}_C}{d\xi} = 0. \tag{2.16}
$$

Thus, by applying (2.16) to (2.15) we obtain

$$
\begin{aligned}
& \partial_t W_G + \partial_x f(t,x,W_G) - g(t,x,W_G) \\
& = \frac{1}{t}\Big(\frac{\partial f}{\partial u}(t,x,W_G) - \frac{\partial f}{\partial u}(0,0,\widetilde{W}_C)\Big)\cdot\frac{d\widetilde{W}_C}{d\xi} \\
& \quad + (\frac{\partial f}{\partial u}(t,x,W_G) - \xi I)\frac{\partial q}{\partial u}(0,0,\widetilde{W}_C)\cdot\frac{d\widetilde{W}_C}{d\xi} + q(0,0,\widetilde{W}_C) - q(t,x,W_G).
\end{aligned} \tag{2.17}
$$

Next, we multiply (2.17) by $\theta(t,x)$ and integrate the equation over the region of i-rarefaction wave: $t < s$ and $\frac{x}{t} \in [\sigma_i^-, \sigma_i^+]$. Due to the Lipschitz continuity of $\frac{\partial f}{\partial u}$ and the fact that $\frac{\partial f}{\partial u}$, $\frac{\partial q}{\partial u}$, $\frac{d\widetilde{W}_C}{d\xi}$ and q remain bounded in $[0,s]\times[-r,r]$, the right hand side of (2.17) is bounded by $O(1)s^2(s+|u_i-u_{i-1}|)$. Therefore, by (2.17) again, we deduce the estimate

$$\begin{aligned}\Delta_i^2(s,r;\theta) = &\int_{\sigma_i^- s}^{\sigma_i^+ s} W_G(s,x)\theta(s,x)dx \\ &+ \int_0^s (f(t,\sigma_i^+ t, W_G(t,\sigma_i^+ t)) - \sigma_i^+ W_G(t,\sigma_i^+ t))\theta(t,\sigma_i^+ t)dt \\ &- \int_0^s (f(t,\sigma_i^- t, W_G(t,\sigma_i^- t)) - \sigma_i^- W_G(t,\sigma_i^- t))\theta(t,\sigma_i^- t)dt \\ &+ O(1)s^2(s+|u_R-u_L|)\|\theta\|_{\mathcal{C}^0}.\end{aligned} \tag{2.18}$$

Next, note that an i-shock wave satisfies the Rankine-Hugoniot condition

$$f(0,0,u_i) - \sigma_i u_i = f(0,0,u_{i-1}) - \sigma_i u_{i-1},$$

and this implies that the approximate solution $W_G(t,x)$ satisfies

$$\begin{aligned}&\int_0^s [(f(t,\sigma_i t, W_G(t,\sigma_i t+)) - \sigma_i W_G(t,\sigma_i t+))]\theta(t,\sigma_i t)dt \\ &- \int_0^s [(f(t,\sigma_{i-1} t, W_G(t,\sigma_{i-1} t-)) - \sigma_{i-1} W_G(t,\sigma_{i-1} t-))]\theta(t,\sigma_{i-1} t)dt \\ &= O(1)s^2|u_R-u_L|\|\theta\|_{\mathcal{C}^0}\end{aligned} \tag{2.19}$$

where the bound $O(1)$ depends on the Lipschitz constant of f and L^∞-norm of q. Finally, by the estimates (2.9), (2.12)-(2.14) and (2.18)-(2.19), we obtain

$$\begin{aligned}\Delta(s,t;\theta) = &\sum_{i=0}^{p} \Delta_i^1(s,t;\theta) + \sum_{\substack{i-rare.\\ waves}} \Delta_i^2(s,t;\theta) \\ = &\int_{-r}^{\sigma_i^- s} W_G(s,x)\theta(s,x)dx + \sum_{i=1}^{p-1}\int_{\sigma_i^+ s}^{\sigma_{i+1}^- s} W_G(s,x)\theta(s,x)dx \\ &+ \sum_{\substack{i-rare.\\ waves}} \int_{\sigma_i^- s}^{\sigma_i^+ s} W_G(s,x)\theta(s,x)dx + \int_{\sigma_p^+ s}^{r} W_G(s,x)\theta(s,x)dx \\ &- \int_{-r}^{0} W_G(0,x)\theta(0,x)dx - \int_0^r W_G(0,x)\theta(0,x)dx \\ &+ \int_0^s f(t,r,W_G(t,r))\theta(t,r)dt - \int_0^s f(t,-r,W_G(t,-r))\theta(t,-r)dt \\ &+ O(1)(s^2+r^2)(s+r+|u_R-u_L|)\ ||\theta||_{\mathcal{C}^1},\end{aligned}$$

which leads to (2.8) and completes the proof. ∎

3. Wave interaction estimates

In this section we study the nonlinear interaction of waves issuing from two Riemann solutions and we derive estimates on the wave strengths.

We emphasize that the generalized Riemann solution, nor the approximate solution $W_G(t,x)$ of the generalized Riemann problem $R_G(u_L,u_R;t_0,x_0)$ is not self-similar. The solution does not consist of regions of constant value separated by straight lines. We thus should be careful in defining the wave strengths In fact, we still define here the wave strengths by using the underlying, classical Riemann solution $W_C(t,x)$. We will see later that this strategy is accurate enough and that the discrepancy in total variation between $W_G(t,x)$ and $W_C(t,x)$ on each time step is uniformly small (Cf. Section 4) when our Glimm scheme is applied to the problem (1.1), (1.2). The same observation applies to the potential of wave interaction to be introduced later.

In the rest of the section, all waves are considered as waves from some classical Riemann problem unless specified otherwise. We say that an i-wave and a j-wave approach each other (or interact in the future) if either $i > j$, or else $i = j$ and at least one of two waves is a shock wave. Suppose there are two solutions from different classical Riemann problems with strengths denoted by $\alpha = (\alpha_i,\dots,\alpha_p)$ and $\beta = (\beta_i,\dots,\beta_p)$, then the *wave interaction potential* associated these two solutions is defined by

$$D(\alpha,\beta) := \sum_{(i,j)} |\alpha_i\beta_j|, \tag{3.1}$$

where the notation (i,j) under the summation sign indicates an i-wave in one solution approaching a j-wave in the other solution, and the summation is on all approaching waves; also α_i or β_i is negative when $i = j$. In addition, given a $(u_L,u_R;t_0,x_0) \in \mathcal{U}\times\mathcal{U}\times\mathbb{R}_+\times\mathbb{R}$, the wave strengths in $R_C(u_L,u_R;t_0,x_0)$ are denoted by $\varepsilon(u_L,u_R;t_0,x_0)$.

We first recall:

Lemma 3.1. (Glimm) *1) Given a (t_0,x_0) in $\mathbb{R}_+\times\mathbb{R}$ and u_L, u_M, u_R in $\mathcal{U}$, we have*

$$|\gamma - (\alpha+\beta)| = O(1)D(\alpha,\beta) \tag{3.2}$$

where

$$\alpha = \varepsilon(u_L,u_M;t_0,x_0), \quad \beta = \varepsilon(u_M,u_R;t_0,x_0), \quad \gamma = \varepsilon(u_L,u_R;t_0,x_0). \tag{3.3}$$

2) Let v_L, v_R be two constant states in $\mathcal{U}$, then

$$D(\gamma,\delta) = D(\alpha,\delta) + D(\beta,\delta) + O(1)|\delta|D(\alpha,\beta), \tag{3.4}$$

and

$$D(\delta,\gamma) = D(\delta,\alpha) + D(\delta,\beta) + O(1)|\delta| D(\alpha,\beta)$$

where α, β *and* γ *are given in* (3.3)*, and* δ *is given by* $\delta = \varepsilon(v_L, v_R; t_0, x_0)$*.*

The following lemma describes the dependence of the wave strengths and potential $D(\cdot,\cdot)$ with respect to their arguments. We introduce the following "local norm" of a given function $\varphi(t,x,u)$

$$N^{x_1,x_2}_{t_1,t_2}(\varphi) = \sup\Big\{ |\varphi(t,x,u)| \,;\, t \in [t_1,t_2],\ x \in [x_1,x_2],\ u \in \mathcal{U} \Big\}, \tag{3.5}$$

where the supremum is taken over any function $u \in \mathcal{U}$ and $(t,x) \in [t_1,t_2] \times [x_1,x_2]$.

Lemma 3.2. *1) The wave strength* $\varepsilon = (\varepsilon_i)_{1\le i\le p} : \mathcal{U} \times \mathcal{U} \times \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}^p$ *is a* $\mathcal{C}^2$ *vector function of its arguments. Furthermore, for any* $(u_L,u_R), (u'_L,u'_R)$ *in* $\mathcal{U} \times \mathcal{U}$ *and any* $(t_0,x_0), (t'_0,x'_0)$ *in* $\mathbb{R}_+ \times \mathbb{R}$*, we have*

$$\begin{aligned} |\alpha' - \alpha| \ &= \ O(1)|\alpha|(|u'_L - u_L| + |u'_R - u_R| + C^0_1|t'_0 - t_0| + C^0_2|x'_0 - x_0|) \\ &\quad + O(1)|(u'_R - u'_L) - (u_R - u_L)| \end{aligned} \tag{3.6}$$

where

$$\alpha = \varepsilon(u_L, u_R; t_0, x_0), \quad \alpha' = \varepsilon(u'_L, u'_R; t'_0, x'_0), \tag{3.7}$$

and the constants C^0_1 *and* C^0_2 *are given by*

$$C^0_1 := N^{x_0,x'_0}_{t_0,t'_0}(\frac{\partial^2 A}{\partial t \partial u}), \qquad C^0_2 := N^{x_0,x'_0}_{t_0,t'_0}(\frac{\partial^2 A}{\partial x \partial u}). \tag{3.8}$$

2) For given $(u_L,u_R), (v_L,v_R), (u'_L,u'_R), (v'_L,v'_R)$ *in* $\mathcal{U}\times\mathcal{U}$ *and* $(t_1,x_1), (t_2,x_2),$ $(t'_1,x'_1), (t'_2,x'_2)$ *in* $\mathbb{R}_+ \times \mathbb{R}$*, we have*

$$\begin{aligned} D(\alpha',\beta') = D(\alpha,\beta) &+ O(1)|\alpha||(v'_R - v'_L) - (v_R - v_L)| \\ &+ O(1)|\beta||(u'_R - u'_L) - (u_R - u_L)| \\ &+ O(1)|\alpha||\beta|\Big(|u'_L - u_L| + |u'_R - u_R| + |v'_L - v_L| + |v'_R - v_R|\Big) \\ &+ O(1)|\alpha||\beta| \sum_{m=1,2} \{C^m_1|t'_m - t_m| + C^m_2|x'_m - x_m|\} \\ &+ O(1)|(u'_R - u'_L) - (u_R - u_L)| \cdot |(v'_R - v'_L) - (v_R - v_L)| \end{aligned} \tag{3.9}$$

where

$$\begin{aligned} &\alpha = \varepsilon(u_L, u_R; t_1, x_1),\ \beta = \varepsilon(v_L, v_R; t_2, x_2), \\ &\alpha' = \varepsilon(u'_L, u'_R; t'_1, x'_1),\ \beta' = \varepsilon(v'_L, v'_R; t'_2, x'_2), \end{aligned} \tag{3.10}$$

and the constants C^m_1, C^m_2 *are defined by*

$$C^m_1 := N^{x_m,x'_m}_{t_m,t'_m}(\frac{\partial^2 A}{\partial t \partial u}), \quad C^m_2 := N^{x_m,x'_m}_{t_m,t'_m}(\frac{\partial^2 A}{\partial x \partial u}),\ m = 1,2. \tag{3.11}$$

Proof. The regularity of functions ε_i, $i = 1, 2, \dots, p$, is a consequence of smoothness of the flux function f and the result of [15]. Moreover, the functions $\frac{\partial^2 \varepsilon_i}{\partial t \partial u_R}$ and $\frac{\partial^2 \varepsilon_i}{\partial x \partial u_R}$ are bounded if $\frac{\partial^2 A}{\partial t \partial u}$, $\frac{\partial^2 A}{\partial x \partial u}$ are bounded.

To show (3.6), we note that $\varepsilon_i(u_L, u_R; t_0, x_0) = 0$ when $u_R = u_L$. Then, by the regularity of ε_i, $i = 1, 2, \dots, p$, we can express $\varepsilon_i(u_L, u_R; t_0, x_0)$, $\varepsilon_i(u'_L, u'_R; t'_0, x'_0)$ as

$$\varepsilon_i(u_L, u_R; t_0, x_0) = \int_0^1 \frac{\partial \varepsilon_i}{\partial u_R}(u_L, (1-\tau)u_L + \tau u_R; t_0, x_0) d\tau \cdot (u_R - u_L),$$

$$\varepsilon_i(u'_L, u'_R; t'_0, x'_0) = \int_0^1 \frac{\partial \varepsilon_i}{\partial u_R}(u'_L, (1-\tau)u'_L + \tau u'_R; t'_0, x'_0) d\tau \cdot (u'_R - u'_L).$$

Applying the definition of $\{C_j^0 : j = 1, 2.\}$ in (3.8) and the norm in (3.5), we obtain

$$\begin{aligned}
&\varepsilon_i(u'_L, u'_R; t'_0, x'_0) - \varepsilon_i(u_L, u_R; t_0, x_0) \\
= &\int_0^1 (\frac{\partial \varepsilon_i}{\partial u_R}(u'_L, (1-\tau)u'_L + \tau u'_R; t'_0, x'_0) - \frac{\partial \varepsilon_i}{\partial u_R}(u_L, (1-\tau)u_L + \tau u_R; t_0, x_0)) d\tau \cdot (u_R - u_L) \\
&+ \int_0^1 \frac{\partial \varepsilon_i}{\partial u_R}(u'_L, (1-\tau)u'_L + \tau u'_R; t'_0, x'_0) d\tau \cdot ((u'_R - u'_L) - (u_R - u_L)) \\
= &O(1)\{|u'_L - u_L| + |u'_R - u_R| + C_1^0|t'_0 - t_0| + C_2^0|x'_0 - x_0|\}|u_R - u_L| \\
&+ O(1)|(u'_R - u'_L) - (u_R - u_L)|,
\end{aligned}$$

Therefore, by the observation of (3.7) and the fact that

$$|u_R - u_L| = O(1)|\varepsilon(u_L, u_R; t_0, x_0)| = O(1)|\alpha|,$$

we obtain (3.6).

Next we derive (3.9). By applying (3.6) directly, we have

$$\begin{aligned}
\alpha'_i &= \alpha_i + O(1)|\alpha|\{|u'_L - u_L| + |u'_R - u_R| + C_1^1|t'_1 - t_1| + C_2^1|x'_1 - x_1|\} \\
&\quad + O(1)|(u'_R - u'_L) - (u_R - u_L)|,
\end{aligned}$$

$$\begin{aligned}
\beta'_j &= \beta_j + O(1)|\beta|\{|v'_L - v_L| + |v'_R - v_R| + C_1^2|t'_2 - t_2| + C_2^2|x'_2 - x_2|\} \\
&\quad + O(1)|(v'_R - v'_L) - (v_R - v_L)|
\end{aligned}$$

for $i,\ j = 1, 2, \dots, p$ where the constants $\{C_j^m : j, m = 1, 2.\}$ are given in (3.11) and (3.5). We define $A := \{|u'_L - u_L| + |u'_R - u_R| + C_1^1|t'_1 - t_1| + C_2^1|x'_1 - x_1|\}$ and $B := \{|v'_L - v_L| + |v'_R - v_R| + C_1^2|t'_2 - t_2| + C_2^2|x'_2 - x_2|\}$. Then by multiplying two previous equations together and using the fact that A, B are of order $O(1)$ for $(u_L, u_R), (u'_L, u'_R) \in \mathcal{U} \times \mathcal{U}$, we obtain

$$\begin{aligned} \alpha'_i \beta'_j &= \alpha_i \beta_j + O(1)|\alpha||\beta|(A+B) + O(1)|\alpha||(v'_R - v'_L) - (v_R - v_L)| \\ &\quad + O(1)|\beta||(u'_R - u'_L) - (u_R - u_L)| \\ &\quad + O(1)|(u'_R - u'_L) - (u_R - u_L)| \cdot |(v'_R - v'_L) - (v_R - v_L)|, \end{aligned}$$

$i, j = 1, 2, \ldots, p$. Summing up previous equations for $i,\ j = 1, 2, \ldots, p$, we obtain (3.9). The proof is completed. $\square$

Using Lemmas 3.1 and 3.2, we obtain wave interaction estimates –which can be interpreted as a generalized version of [10].

Proposition 3.3. *1) Suppose that s, r are two positive numbers and (t_0, x_0) is in $\mathbb{R}_+ \times \mathbb{R}$. Also assume that u_L, u_M, u_R, $u_L + \mu_L$, $u_R + \mu_R$ are constant states in $\mathcal{U}$ and α, β and γ are the wave strengths of solutions of three classical Riemann problems $R_C(u_L, u_M; t_0, x_0 - r)$, $R_C(u_M, u_R; t_0, x_0 + r)$ and $R_C(u_L + \mu_L, u_R + \mu_R; t_0 + s, x_0)$, i.e.,*

$$\begin{aligned} &\alpha = \varepsilon(u_L, u_M; t_0, x_0 - r), \quad \beta = \varepsilon(u_M, u_R; t_0, x_0 + r), \\ &\gamma = \varepsilon(u_L + \mu_L, u_R + \mu_R; t_0 + s, x_0). \end{aligned} \tag{3.12}$$

Then we have

$$\begin{aligned} |\gamma| &= |\alpha| + |\beta| + O(1)D(\alpha, \beta) \\ &\quad + O(1)(|\alpha| + |\beta|)(|\mu_L| + |\mu_R| + C_1 s + C_2 r) \\ &\quad + O(1)|\mu_R - \mu_L| \end{aligned} \tag{3.13}$$

where constants C_1 and C_2 are defined by

$$C_1 := N^{x_0 x_0}_{t_0, t_0+s}\Big(\frac{\partial^2 A}{\partial t \partial u}\Big), \quad C_2 := N^{x_0 - r, x_0 + r}_{t_0, t_0}\Big(\frac{\partial^2 A}{\partial x \partial u}\Big). \tag{3.14}$$

2) Let α, β, γ be the wave strengths as described in (3.12)*. Also, for a given (v_L, v_R) in $\mathcal{U} \times \mathcal{U}$ and (t_1, x_1) in $\mathbb{R}_+ \times \mathbb{R}$, we define $\delta = \varepsilon(v_L, v_R; t_1, x_1)$. Then*

$$\begin{aligned} D(\gamma, \delta) = D(\alpha, \delta) + D(\beta, \delta) + O(1)|\delta| D(\alpha, \beta) + O(1)|\delta||\mu_R - \mu_L| \\ + O(1)|\delta|(|\alpha| + |\beta|)(|\mu_L| + |\mu_R| + C_1 s + C_2 r), \end{aligned} \tag{3.15}$$

and

$$\begin{aligned} D(\delta, \gamma) = D(\delta, \alpha) + D(\delta, \beta) + O(1)|\delta| D(\alpha, \beta) + O(1)|\delta||\mu_R - \mu_L| \\ + O(1)|\delta|(|\alpha| + |\beta|)(|\mu_L| + |\mu_R| + C_1 s + C_2 r) \end{aligned} \tag{3.16}$$

where constants C_1 and C_2 are given in (3.14)*.*

Proof. By the definition of γ in (3.12) and Lemma 3.2 with $u'_L = u_L + \mu_L$, $u'_R = u_R + \mu_R$, $t' = t_0 + s$, $x'_0 = x_0$, we obtain

$$\begin{aligned}\gamma = \; & \varepsilon(u_L, u_R; t_0, x_0) + O(1)|\varepsilon(u_L, u_R; t_0, x_0)| \Big(|\mu_L| + |\mu_R| + C_1\, s\Big) \\ & + O(1)\,|\mu_R - \mu_L|\end{aligned} \tag{3.17}$$

where constant C_1 is given in (3.14). Similarly, by Lemma 3.2 we have

$$\varepsilon(u_L, u_M; t_0, x_0) = \alpha + O(1)C_2|\alpha|r, \tag{3.18}$$

$$\varepsilon(u_M, u_R; t_0, x_0) = \beta + O(1)C_2|\beta|r. \tag{3.19}$$

On the other hand, Glimm's interaction estimates (3.2), (3.3) lead to

$$\begin{aligned}\varepsilon(u_L, u_R; t_0, x_0) = \; & \varepsilon(u_L, u_M; t_0, x_0) + \varepsilon(u_M, u_R; t_0, x_0) \\ & + O(1)D(\varepsilon(u_L, u_M; t_0, x_0), \varepsilon(u_M, u_R; t_0, x_0)).\end{aligned} \tag{3.20}$$

Also, by (3.9)-(3.11) with $\alpha' = \varepsilon(u_L, u_M; t_0, x_0)$ and
$\beta' = \varepsilon(u_M, u_R; t_0, x_0)$, we obtain

$$D(\varepsilon(u_L, u_M; t_0, x_0), \varepsilon(u_M, u_R; t_0, x_0)) = D(\alpha, \beta) + O(1)|\alpha||\beta|C_2 r. \tag{3.21}$$

Then, from (3.17)-(3.21) it follows that

$$\begin{aligned}|\gamma| \;&=\; |\alpha| + |\beta| + O(1)D(\alpha, \beta) + O(1)(|\alpha| + |\beta| + |\alpha||\beta|)C_2 r \\ & \quad + O(1)|\varepsilon(u_L, u_R; t_0, x_0)|(|\mu_L| + |\mu_R| + C_1 s) + O(1)|\mu_R - \mu_L| \\ &=\; |\alpha| + |\beta| + O(1)D(\alpha, \beta) + O(1)(|\alpha| + |\beta|)C_2 r \\ & \quad + O(1)|\varepsilon(u_L, u_R; t_0, x_0)|(|\mu_L| + |\mu_R| + C_1 s) + O(1)|\mu_R - \mu_L|.\end{aligned} \tag{3.22}$$

Also, we see that estimates (3.20) and (3.21) yield

$$\begin{aligned}|\varepsilon(u_L, u_R; t_0, x_0)| \;&= |\varepsilon(u_L, u_M; t_0, x_0)| + |\varepsilon(u_M, u_R; t_0, x_0)| \\ & \quad + O(1)D(\varepsilon(u_L, u_M; t_0, x_0), \varepsilon(u_M, u_R; t_0, x_0)) \\ &= |\alpha| + |\beta| + O(1)D(\alpha, \beta) + O(1)\left(|\alpha| + |\beta|\right) C_2 r \\ & \quad + O(1)|\alpha||\beta|C_2 r \\ &= (|\alpha| + |\beta|)(1 + O(1)C_2 r) + O(1)D(\alpha, \beta) + O(1)|\alpha||\beta|C_2 r,\end{aligned}$$

which in particular implies that

$$|\varepsilon(u_L, u_R; t_0, x_0)| = O(1)(|\alpha| + |\beta|). \tag{3.23}$$

Therefore, combining (3.22) with (3.23), we obtain (3.13).

Next we derive (3.15). The proof of (3.16) is similar, and is omitted. By the estimate (3.4) we see that

$$\begin{aligned} D(\varepsilon(u_L, u_R; t_0, x_0), \delta) &= D(\varepsilon(u_L, u_M; t_0, x_0), \delta) + D(\varepsilon(u_M, u_R; t_0, x_0), \delta) \\ &\quad + O(1)|\delta| D(\varepsilon(u_L, u_M; t_0, x_0), \varepsilon(u_M, u_R; t_0, x_0)). \end{aligned} \tag{3.24}$$

On the other hand, estimate (3.9) yields

$$\begin{aligned} D(\gamma, \delta) &= D(\varepsilon(u_L, u_R; t_0, x_0), \delta) + O(1)|\delta||\mu_R - \mu_L| \\ &\quad + O(1)|\varepsilon(u_L, u_R; t_0, x_0)||\delta|(|\mu_L| + |\mu_R| + C_1 s), \end{aligned} \tag{3.25}$$

$$D(\varepsilon(u_L, u_M; t_0, x_0), \delta) = D(\alpha, \delta) + O(1)|\alpha||\delta| C_2 r, \tag{3.26}$$

$$D(\varepsilon(u_M, u_R; t_0, x_0), \delta) = D(\beta, \delta) + O(1)|\beta||\delta| C_2 r, \tag{3.27}$$

and

$$D(\varepsilon(u_L, u_M; t_0, x_0), \varepsilon(u_M, u_R; t_0, x_0)) = D(\alpha, \beta) + O(1)|\alpha||\beta| C_2 r. \tag{3.28}$$

Thus, by applying (3.23), (3.25)-(3.28) to (3.24), we obtain the estimate (3.15). The proof is completed. $\square$

We just showed in Proposition 3.3 that Glimm's interaction estimates (Lemma 3.1) remain valid for the quasilinear hyperbolic system (1.1) up to certain error terms. The following immediate consequence of Proposition 3.3 will be the key to the forthcoming stability result.

Corollary 3.4 *Following the notations and assumptions in Proposition 3.3 and letting*

$$\mu_L := -sq(t_0 + s, x_0, u_L), \;\; \mu_R := -sq(t_0 + s, x_0, u_R)$$

in (3.14), we have

$$\begin{aligned} |\gamma| &= |\alpha| + |\beta| + O(1) D(\alpha, \beta) \\ &\quad + O(1)(|\alpha| + |\beta|)\Big((C_1 + C_3 + C_4)\, s + C_2\, r\Big), \end{aligned} \tag{3.29}$$

$$\begin{aligned} D(\delta, \gamma) &= D(\delta, \alpha) + D(\delta, \beta) + O(1)|\delta| D(\alpha, \beta) \\ &\quad + O(1)|\delta|(|\alpha| + |\beta|)\Big((C_1 + C_3 + C_4)\, s + C_2\, r\Big), \end{aligned} \tag{3.30}$$

$$\begin{aligned} D(\gamma, \delta) &= D(\alpha, \delta) + D(\beta, \delta) + O(1)|\delta| D(\alpha, \beta) \\ &\quad + O(1)|\delta|(|\alpha| + |\beta|)\{(C_1 + C_3 + C_4)s + C_2 r\} \end{aligned} \tag{3.31}$$

where constants C_1, C_2 are given in (3.14) *and C_3, C_4 are given by*

$$C_3 := N^{x_0 x_0}_{t_0, t_0+s}(q(t, x, u)), \quad C_4 := N^{x_0 x_0}_{t_0, t_0+s}\left(\frac{\partial q}{\partial u}(t, x, u)\right). \tag{3.32}$$

Proof. By the observation of (3.23) we obtain

$$
\begin{aligned}
|\mu_R - \mu_L| &= s|q(t_0+s, x_0, u_R) - q(t_0+s, x_0, u_L)| \\
&= s\frac{\partial q}{\partial u}(t_0+s, x_0, \bar{u}) \cdot |u_R - u_L| \\
&= O(1)C_4 s|\varepsilon(u_L, u_R; t_0, x_0)| \\
&= O(1)(|\alpha| + |\beta|)C_4 s
\end{aligned}
\tag{3.33}
$$

where $\bar{u} \in \mathcal{U}$ and C_4 is given in (3.32). Therefore, by combining (3.33) with the result of Proposition 3.3, we obtain (3.29)-(3.31) . The proof is completed. □

4. Stability of the generalized Glimm method

We are in position to introduce our version of Glimm scheme for the approximation of the quasilinear system (1.1). Then we rely on the wave interaction estimates in Section 3 and prove a stability result.

The approximate solution to the Cauchy problem (1.1), (1.2) is defined as follows. Given two positive constants s and r satisfying the C-F-L condition (2.7), we introduce the constant

$$\lambda_* := \frac{r}{s}. \tag{4.1}$$

Let also $a = \{a_k : a_k \in (-1,1), k \in \mathbb{N}\}$ be an equidistributed sequence. We divide the (t,x) plane into

$$t_k = ks,\ x_h = hr, \quad k = 0, 1, 2, \ldots,\ h \in \mathbb{Z}. \tag{4.2}$$

Next, we construct an approximate solution $u_r(t,x)$ of the problem (1.1), (1.2) in the following way. First, the initial data $u_0(x)$ is approximated by a piecewise constant function

$$u_r(0,x) = u_0(hr), \quad x \in [(h-1)r, (h+1)r),\ h \text{ is odd}. \tag{4.3}$$

Then, within domain $0 \le t < s$, we construct an approximate solution $W_G(t,x)$ for each generalized Riemann problem with initial data $u_r(0,x)$ to obtain $u_r(t,x)$ in region $\{(t,x); 0 \le t < s\}$. If $u_r(t,x)$ has been constructed for $t < ks$, $k \in \mathbb{N}$, we set

$$u_r(ks,x) \ := u_r(ks-, (h+a_k)r) \tag{4.4}$$

for $x \in [(h-1)r, (h+1)r)$, $k+h$ is odd. Again, we solve the generalized Riemann problems with initial data $u_r(ks,x)$ given in (4.4) to construct $u_r(t,x)$ within region $\{(t,x); ks \le t < (k+1)s\}$. Following the process (4.3), (4.4) consecutively, we then construct our approximate solution $u_r(t,x)$ of (1.1), (1.2). In other words, the approximate solution to the problem (1.1), (1.2) generated by the generalized Glimm scheme is given by

$$u_r(t,x) = W_G(t,x; u_r(ks,(h-1)r), u_r(ks,(h+1)r); ks, hr) \tag{4.5}$$

for $(t,x) \in [ks,(k+1)s) \times [(h-1)r,(h+1)r)$, $k+h$ is even.

Next we study the stability of $u_r(t,x)$ in L^∞ and BV norms. This requires the description of mesh points, mesh curves and immediate successors beforehand. Recall that the values of $u_r(t,x)$ on $t = ks$ are determined by the values of $u_r(t,x)$ at points $\{(ks-,(h+a_k)r);\ h \in \mathbb{Z},\ k+h \text{ is odd}\}$, we call these points $\{(ks,(h+a_k)r) : k = 0,1,2,\cdots,\ h \in \mathbb{Z},\ k+h \text{ is odd}\}$ the *mesh points* of approximate solution $u_r(t,x)$. We obtain a set of diamond regions by connecting all mesh points with segments. An unbounded piecewise linear curve I is called a *mesh curve* if I lies on the boundaries of those diamond regions. Suppose I is a mesh curve, then I divides the (t,x) plane into I^+ and I^- regions, such that I^- contains $t=0$. We say two mesh curves $I_1 > I_2$ (I_1 is a *successor* of I_2) if every point of I_1 is either on I_2 or contained in I_2^+. And, I_1 is an *immediate successor* of I_2 if $I_1 > I_2$ and every mesh point of I_1 except one is on I_2. Note that the difference between I_1 and I_2 is determined by a diamond region if one is an immediate successor of the other.

Next, to simplify the notations, we set $u_{k,h} := u_r(ks,hr)$ when $k+h$ is odd. By the observation of (2.4) and (4.5), we have

$$u_{k,h} = \tilde{u}_{k,h} + s\,q((k-1)s, hr, \tilde{u}_{k,h}), \quad k+h \text{ is odd},$$

where $\tilde{u}_{k,h}$ is the value of $R_C(u_{k-1,h-1}, u_{k-1,h+1}; (k-1)s, hr)$ at $(ks-,(h+a_k)r)$, i.e.,

$$\tilde{u}_{k,h} = W_C(a_k\frac{r}{s}; u_{k-1,h-1}, u_{k-1,h+1}; (k-1)s, hr)$$

with the function W_C given in Section 2. Next, given a pair (k_0,h_0), k_0+h_0 is even, we note that the $(t,x)-$plan consists of the diamond regions Γ_{k_0,h_0} with center (k_0s, h_0r) and vertices (mesh points)

$$\begin{aligned} &S := ((k_0-1)s,(h_0+a_{k_0-1})r),\ W := (k_0s,(h_0-1+a_{k_0})r),\\ &E := (k_0s,(h_0+1+a_{k_0})r),\ N := ((k_0+1)s,(h_0+a_{k_0+1})r) \end{aligned} \tag{4.6}$$

(see Figure 4.1). We set

$$u_S := u_{k_0-1,h_0},\ u_W := u_{k_0,h_0-1},\ u_E := u_{k_0,h_0+1},\ u_N := u_{k_0+1,h_0}, \tag{4.7}$$

and

$$\tilde{u}_S := \tilde{u}_{k_0-1,h_0},\ \tilde{u}_W := \tilde{u}_{k_0,h_0-1},\ \tilde{u}_E := \tilde{u}_{k_0,h_0+1},\ \tilde{u}_N := \tilde{u}_{k_0+1,h_0}. \tag{4.8}$$

Note that u_W and u_E are the states in $R_G((k_0-1)s,(h_0-1)r)$ and $R_G((k_0-1)s,(h_0+1)r)$ respectively, i.e.,

$$u_W = \tilde{u}_W + s\,q((k_0-1)s,(h_0-1)r,\tilde{u}_W),$$

$$u_E = \tilde{u}_E + s\,q((k_0-1)s,(h_0+1)r,\tilde{u}_E).$$

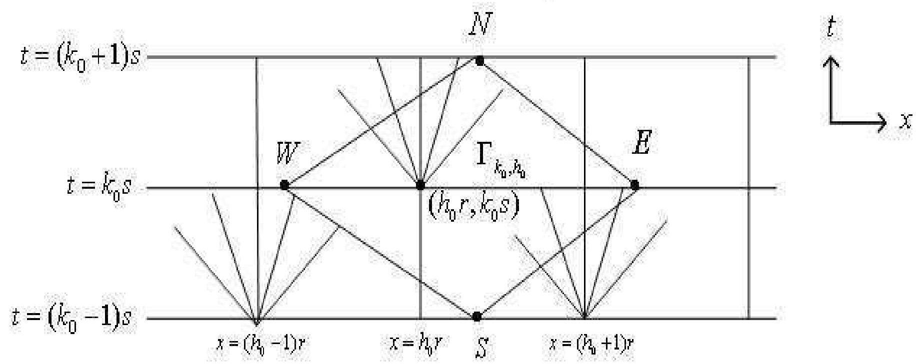

Figure 4.1 : Diamond region Γ_{k_0,h_0}

Now we define the *strengths* of waves in $u_r(t,x)$. However, the set up for the waves strengths of $u_r(t,x)$ becomes crucial due to the lack of self-similarity of approximate solution $W_G(t,x)$, the strengths of waves in $W_G(t,x)$ can not be defined in the traditional way as described in [15]. To overcome the difficulty, we first solve the associated classical Riemann problems with the initial data $\{u_r(ks-,(h+a_k)r);\ x\in[(h-1)r,(h+1)r),\ k+h \text{ is odd}\}$ (see (4.4)) within each time step. So we construct a new function $\tilde{u}_r(t,x)$ defined on $\mathbb{R}_+\times\mathbb{R}$. Then we define the strengths of approximate waves in $u_r(t,x)$ based on classical waves in $\tilde{u}_r(t,x)$. More precisely, given a wave $(u_{i-1}(t),u_i(t))$ in $u_r(t,x)$, there exist two corresponding constant states u_{i-1}, u_i and a classical Riemann wave (u_{i-1},u_i) with strength $\varepsilon(u_{i-1},u_i)$ in $\tilde{u}_r(t,x)$, then the strength of $(u_{i-1}(t),u_i(t))$ is defined as $\varepsilon(u_{i-1},u_i)$.

Next, we show that, under the condition that the $L^1(\mathbb{R}_+\times\mathbb{R})$-norms of q and $\frac{\partial q}{\partial u}$ are small, the sum of strengths for waves in $u_r(t,x)$ crossing mesh curve J can be regarded as an equivalent norm for the total variation of $u_r(t,x)$ on J. By the fact that the term $|\varepsilon(u_{i-1},u_i)|$ is equivalent to the total variation of (u_{i-1},u_i) for any classical Riemann wave (u_{i-1},u_i), it is equivalent to show that the total variation of $u_r(t,x)$ on J is equivalent to the total variation of $\tilde{u}_r(t,x)$ on J. To show this, let J_k be a mesh curve lying within k-th time level $\{(t,x);\ ks\le t<(k+1)s\}$, and let $TV(u_r(t,x),J_k)$, $TV(\tilde{u}_r(t,x),J_k)$ denote the total variations of $u_r(t,x)$, $\tilde{u}_r(t,x)$ on J_k respectively. Suppose there is a wave $(u_{i-1}(t),u_i(t))$ in $u_r(t,x)$, issued from (ks,ir) and crosses J_k, also (u_{i-1},u_i) is the corresponding classical Riemann wave of $(u_{i-1}(t),u_i(t))$ (so (u_{i-1},u_i) is also issued from (ks, ir) and crosses J_k). If (u_{i-1}, u_i) is a shock wave, then by (2.4) we can easily obtain that

$$\begin{aligned}&|TV((u_{i-1}(t), u_i(t)); J_k) - TV((u_{i-1}, u_i); J_k)|\\ &\leq s\,|\frac{\partial q}{\partial u}(ks, ir, \bar{u}_i)|\,TV((u_{i-1}, u_i); J_k) + s\,\big(|q(t_0, x_0, u_{i-1})| + |q(t_0, x_0, u_i)|\big),\end{aligned}$$

where $\bar{u}_i \in \mathcal{U}$ and $TV((u_{i-1}(t), u_i(t)); J_k)$, $TV((u_{i-1}, u_i); J_k)$ denote the total variations of $(u_{i-1}(t), u_i(t))$, (u_{i-1}, u_i) crossing J_k. Similarly, if $(u_{i-1}, u_i) = \bar{u}_i(\xi)$ is a rarefaction wave with $\xi \in [\xi_1, \xi_2]$, then we obtain

$$\begin{aligned}&|TV((u_{i-1}(t), u_i(t)); J_k) - TV((u_{i-1}, u_i); J_k)|\\ &\leq s\,|\frac{\partial q}{\partial u}(ks, ir, \bar{u}_i(\tilde{\xi})|\,TV((u_{i-1}, u_i); J_k) + s(|q(t_0, x_0, u_{i-1})| + |q(t_0, x_0, u_i)|)\end{aligned}$$

for some $\tilde{\xi} \in [\xi_1, \xi_2]$ and $\bar{u}_i(\tilde{\xi}) \in \mathcal{U}$. Summing up the previous inequalities with respect to the waves crossing J_k we obtain

$$\begin{aligned}&|TV(u_r(J_k)) - TV(\tilde{u}_r(J_k))|\\ &\leq O(s)\ \|\frac{\partial q}{\partial u}\|_{L^1(\mathbb{R}_+\times\mathbb{R})}\ \ TV(\tilde{u}_r(J_k)) + O(s)\,\|q\|_{L^1(\mathbb{R}_+\times\mathbb{R})}\end{aligned}$$

for any mesh curve J_k, and this is enough to imply that the total variations of $u_r(t,x)$ and $\tilde{u}_r(t,x)$ on any mesh curve J_k are equivalent when $\|q\|_{L^1(\mathbb{R}_+\times\mathbb{R})}$ and $\|\frac{\partial q}{\partial u}\|_{L^1(\mathbb{R}_+\times\mathbb{R})}$ are small, we then show the statement.

We note that the waves entering each diamond region may come from two generalized Riemann solutions, we certainly need to know the constant states of corresponding classical Riemann solutions at the left and right vertices of diamond region to calculate those wave strengths separately. We proceed as follows.

First, using the notations in (4.7), (4.8), we define the strength of the waves entering the diamond region Γ_{k_0,h_0}, $k_0 + h_0$ is even, by

$$\begin{aligned}\varepsilon_*(\Gamma_{k_0,h_0}) := &|\varepsilon(\tilde{u}_W, u_S; (k_0-1)s, (h_0-1)r)|\\ &+ |\varepsilon(u_S, \tilde{u}_E; (k_0-1)s, (h_0+1)r)|\end{aligned}$$

and the strength of the waves leaving Γ_{k_0,h_0} by

$$\varepsilon^*(\Gamma_{k_0,h_0}) := |\varepsilon(u_W, \tilde{u}_N; k_0 s, h_0 r)| + |\varepsilon(\tilde{u}_N, u_E; k_0 s, h_0 r)|. \tag{4.9}$$

Since $\tilde{u}_N$ is a constant state in $W_C(u_W, u_E; k_0 s, h_0 r)$, we can write

$$\varepsilon^*(\Gamma_{k_0,h_0}) = |\varepsilon(u_W, u_E; k_0 s, h_0 r)|. \tag{4.10}$$

Next, for k_0+h_0 is even, we let $Q(\Gamma_{k_0,h_0})$ denote the potential of waves interaction in the diamond Γ_{k_0,h_0}, i.e.,

$$Q(\Gamma_{k_0,h_0}) := D(\varepsilon(\tilde{u}_W, u_S, (k_0-1)s; (h_0-1)r), \varepsilon(u_S, \tilde{u}_E, (k_0-1)s, (h_0+1)r))$$

where $D(\cdot,\cdot)$ is defined in (3.1). Given a mesh curve J, we note that there are two types of waves crossing J. The first kind of waves are $(\tilde{u}_{k,h-1}, u_{k-1,h})$, $k+h = even$ (waves of type I), the second type of waves are $(u_{k-1,h}, \tilde{u}_{k,h+1})$, $k+h = even$ (waves of type II). More precisely, waves of type I are either of the form $(\tilde{u}_{k,h-1}, u_{k-1,h})$ entering $\Gamma_{k,h}$ (left in-coming waves of $\Gamma_{k,h}$), or $(\tilde{u}_{k+1,h}, u_{k,h+1})$ leaving $\Gamma_{k,h}$ (right out-going waves of $\Gamma_{k,h}$). Waves of type II are either of the form $(u_{k-1,h}, \tilde{u}_{k,h+1})$ entering $\Gamma_{k,h}$ (right in-coming waves of $\Gamma_{k,h}$), or $(u_{k,h-1}, \tilde{u}_{k+1,h})$ leaving $\Gamma_{k,h}$ (left out-going waves of $\Gamma_{k,h}$), see Figures 4.2 (a), (b). Next we define the linear functional $L(J)$ for the waves in $u_r(t,x)$ crossing mesh curve J by

$$\begin{aligned} L(J) := &\sum_{\text{type I}} |\varepsilon(\tilde{u}_{k,h-1}, u_{k-1,h}; (k-1)s, (h-1)r)| \\ &+ \sum_{\text{type II}} |\varepsilon(u_{k-1,h}, \tilde{u}_{k,h+1}; (k-1)s, (h+1)r)|. \end{aligned} \tag{4.11}$$

From previous analysis, we see that functional $L(J)$ is equivalent to the total variation of $u_r(t,x)$ crossing mesh curve J. Next we define the quadratic functional $Q(J)$ of $u_r(t,x)$ by

$$Q(J) := \sum_{(\alpha,\beta)} D(\alpha,\beta) \tag{4.12}$$

where the notation (α,β) under summation sign denotes a pair of waves α, β crossing J and approach, and $D(\alpha,\beta)$ is given in (3.1). Furthermore, we define the *Glimm functional* $F(J)$ of $u_r(t,x)$ for mesh curve J by

$$F(J) := L(J) + K\,Q(J). \tag{4.13}$$

Our goal is to show that functional F remains uniformly bounded on all mesh curves provided that constant K in (4.13) is sufficiently large, and this leads to the result that functional L can be bounded by a constant times the total variation of initial data $u_0(x)$. To show this, the first step is to estimate the possible changing amount of L and Q when waves pass through one mesh curve and into an immediate successor. The estimates of changing amounts of L and Q are stated as follows.

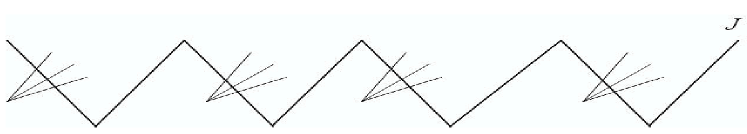

Figure 4.2(a) : Waves of type I crossing mesh curve J

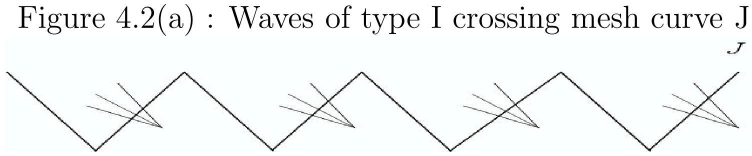

Figure 4.2(b) : Waves of type II crossing mesh curve J

Proposition 4.1. *Given two mesh curves J_1 and J_2 such that J_2 is an immediate successor of J_1, let Γ_{k_0,h_0} denote the diamond region bounded by J_1 and J_2. Then functionals L and Q satisfy*

$$L(J_2) - L(J_1) = O(1)\{Q(\Gamma_{k_0,h_0}) + \varepsilon_*(\Gamma_{k_0,h_0})(C_1^0 + \lambda_* C_2^0 + C_3^0 + C_4^0)s\}, \tag{4.14}$$

$$\begin{aligned} Q(J_2) - Q(J_1) = & -Q(\Gamma_{k_0,h_0}) + O(1)L(J_1)Q(\Gamma_{k_0,h_0}) \\ & + O(1)L(J_1)\varepsilon_*(\Gamma_{k_0,h_0})(C_1^0 + \lambda_* C_2^0 + C_3^0 + C_4^0)s \end{aligned} \tag{4.15}$$

where constants λ_* is defined in (4.1) *and $C_j^0, 1 \le j \le 4$, are given by*

$$C_1^0 := N_{(k_0-1)s,k_0 s}^{h_0 r,h_0 r}(\frac{\partial^2 A}{\partial t \partial u}), \qquad C_2^0 := N_{(k_0-1)s,(k_0-1)s}^{(h_0-1)r,(h_0+1)r}(\frac{\partial^2 A}{\partial x \partial u}), \tag{4.16}$$

$$C_3^0 := N_{(k_0-1)s,k_0 s}^{h_0 r,h_0 r}(q(t,x,u)), \qquad C_4^0 := N_{(k_0-1)s,k_0 s}^{h_0 r,h_0 r}(\frac{\partial q}{\partial u}(t,x,u)) \tag{4.17}$$

where N is defined in (3.5). *Note that $\{C_j^0; 1 \le j \le 4\}$ depend on h_0,k_0.*

Proof. Let u_S, u_W, u_E, u_N be the constant states described in (4.6)-(4.7), we first derive (4.14). By the definitions of ε_* and ε^* in (4.9), (4.10) and L in (4.11), we find

$$\begin{aligned} L(J_2) - L(J_1) &= |\varepsilon(u_W, \tilde{u}_N; k_0 s, h_0 r)| + |\varepsilon(\tilde{u}_N, u_E; k_0 s, h_0 r)| \\ &\quad - |\varepsilon(\tilde{u}_W, u_S; (k_0-1)s, (h_0-1)r)| \\ &\quad - |\varepsilon(u_S, \tilde{u}_E; (k_0-1)s, (h_0+1)r)| \\ &= \varepsilon^*(\Delta_{k_0,h_0}) - \varepsilon_*(\Delta_{k_0,h_0}). \end{aligned} \tag{4.18}$$

Next, by applying the definition of λ_* in (4.1) and the estimates (3.29), (3.32) to (4.18) with the choice of $u_L = \tilde{u}_W,\ u_M = u_S,\ u_R = \tilde{u}_E,\ \mu_L = u_W - \tilde{u}_W,\ \mu_R = u_E - \tilde{u}_E$, $t_0 = (k_0-1)s$ and $x_0 = h_0 r$, we obtain

$$\varepsilon^*(\Gamma_{k_0,h_0}) = \varepsilon_*(\Gamma_{k_0,h_0}) + O(1)Q(\Gamma_{k_0,h_0}) + O(1)\varepsilon_*(\Gamma_{k_0,h_0})\Big((C_1^0 + C_3^0 + C_4^0)s + C_2^0 r\Big),$$

and this gives (4.14).

To prove (4.15), we define several notations for the rest of the section. First, given (k,h), $k+h=$ even, we let vector $\varepsilon_{k-1,h-1/2}$ denote the strength of waves issued from $((k-1)s,(h-1)r)$ entering $\Gamma_{k,h}$, and let vector $\varepsilon_{k-1,h+1/2}$ denote the strength of waves issued from $((k-1)s,(h+1)r)$ entering $\Gamma_{k,h}$. More precisely, the vector $\varepsilon_{k-1,h-1/2}$ measures the strength of waves of type I entering $\Gamma_{k,h}$ and $\varepsilon_{k-1,h+1/2}$ measures the strength of waves of type II entering $\Gamma_{k,h}$. Next, given a mesh curve J, let $J_{[h-1,h]}$ ($J_{[h,h+1]}$ respectively) denote the segment of J in

$\mathbb{R}_+ \times [(h-1)r, hr]$ ($\mathbb{R}_+ \times [hr, (h+1)r]$). Then we define vectors $\varepsilon_{J,h-1/2}$, $\varepsilon_{J,h+1/2}$ as the strengths of waves crossing $J_{[(h-1),h]}$, $J_{[h,h+1]}$ respectively. We will drop the sign J in $\varepsilon_{J,h-1/2}$ and $\varepsilon_{J,h+1/2}$ when J is specified. We also set

$$\varepsilon_{W,S} := \varepsilon_{h_0-1/2}, \quad \varepsilon_{S,E} := \varepsilon_{h_0+1/2},$$

$$\varepsilon_{W,N} := \varepsilon(u_W, \tilde{u}_N; k_0 s, h_0 r), \quad \varepsilon_{N,E} := \varepsilon(\tilde{u}_N, u_E; k_0 s, h_0 r).$$

Since J_2 is an immediate successor of J_1, the diamond region bounded by J_1, J_2 can be specified as $\Gamma(k_0, h_0)$ with center $(k_0 s,\ h_0 r)$, and J_1, J_2 coincide outside $\Gamma(k_0, h_0)$. We will also drop the signs J_1, J_2 without confusion. From the definition of Q in (4.12), we have

$$\begin{aligned}
&Q(J_2) - Q(J_1) \\
&= \sum_{h<h_0} \Big(D(\varepsilon_{h-1/2}, \varepsilon_{W,N}) + D(\varepsilon_{h-1/2}, \varepsilon_{N,E}) - D(\varepsilon_{h-1/2}, \varepsilon_{W,S}) - D(\varepsilon_{h-1/2}, \varepsilon_{S,E}) \Big) \\
&+ \sum_{h>h_0+1} \Big(D(\varepsilon_{W,N}, \varepsilon_{h-1/2}) + D(\varepsilon_{N,E}, \varepsilon_{h-1/2}) - D(\varepsilon_{W,S}, \varepsilon_{h-1/2}) - D(\varepsilon_{S,E}, \varepsilon_{h-1/2}) \Big) \\
&+ D(\varepsilon_{W,N}, \varepsilon_{N,E}) - D(\varepsilon_{W,S}, \varepsilon_{S,E}).
\end{aligned}$$

From (3.1) we see that

$$D(\varepsilon_{W,N}, \varepsilon_{N,E}) = 0. \tag{4.19}$$

Also, for any $h \in \mathbb{Z}$ we observe that

$$D(\varepsilon_{h-1/2}, \varepsilon_{W,N}) + D(\varepsilon_{h-1/2}, \varepsilon_{N,E}) = D(\varepsilon_{h-1/2}, \varepsilon_{W,E}) \tag{4.20}$$

for $h < h_0$, and

$$D(\varepsilon_{W,N}, \varepsilon_{h-1/2}) + D(\varepsilon_{N,E}, \varepsilon_{h-1/2}) = D(\varepsilon_{W,E}, \varepsilon_{h-1/2}) \tag{4.21}$$

for $h > h_0 + 1$. Thus, by (4.19)-(4.21) we obtain

$$\begin{aligned}
&Q(J_2) - Q(J_1) \\
&= \sum_{h<h_0} \Big(D(\varepsilon_{h-1/2}, \varepsilon_{W,E}) - D(\varepsilon_{h-1/2}, \varepsilon_{W,S}) - D(\varepsilon_{h-1/2}, \varepsilon_{S,E}) \Big) \\
&\quad + \sum_{h>h_0+1} \Big(D(\varepsilon_{W,E}, \varepsilon_{h-1/2}) - D(\varepsilon_{W,S}, \varepsilon_{h-1/2}) - D(\varepsilon_{S,E}, \varepsilon_{h-1/2}) \Big) \\
&\quad - D(\varepsilon_{W,S}, \varepsilon_{S,E}).
\end{aligned} \tag{4.22}$$

Finally, applying (3.30) and (3.31) to (4.22) and using the fact that $D(\varepsilon_{W,S}, \varepsilon_{S,E}) =$

$Q(\Gamma_{k_0,h_0})$, we obtain

$$\begin{aligned}
&Q(J_2)-Q(J_1)\\
&=-D(\varepsilon_{W,S},\varepsilon_{S,E})+\sum_{\substack{h\in\mathbb{Z}\\ h\neq h_0,h_0+1}}\Big(O(1)|\varepsilon_{h-1/2}|\,D(\varepsilon_{W,S},\varepsilon_{S,E})\\
&\quad+O(1)|\varepsilon_{h-1/2}|(|\varepsilon_{W,S}|+|\varepsilon_{S,E}|)((C_1^0+C_3^0+C_4^0)s+C_2^0r)\Big)\\
&=-Q(\Gamma_{k_0,h_0})+O(1)L(J_1)Q(\Gamma_{k_0,h_0})+O(1)L(J_1)\,\varepsilon_*(\Gamma_{k_0,h_0})((C_1^0+C_3^0+C_4^0)\,s+C_2^0\,r),
\end{aligned}$$

which leads to (4.15). This completes the proof. □

Before stating a crucial technical lemma, let us introduce a notation about mesh curves. We say that a mesh curve J is of the type (k_0,k_0+1) if all the mesh points on J have the form of $\{(ks,(h+a_k)r):\,k=k_0,\ k_0+1\}$.

Lemma 4.2. *Given a positive integer k_0, let J_1 and J_2 be two mesh curves of type (k_0-1,k_0) and (k_0,k_0+1) respectively. We assume that there exists a positive constant M_* such that*

$$L(J_1)\le M_*. \tag{4.23}$$

If M_* is sufficiently small and the constant K in (4.13) is sufficiently large, then the functional F satisfies the following inequality

$$F(J_2)\le F(J_1)+O(1)s\sum_{h_0\in\mathbb{Z}}\varepsilon_*(\Gamma_{k_0,h_0})(C_1^0+\lambda_*C_2^0+C_3^0+C_4^0) \tag{4.24}$$

where the bound $O(1)$ depends on M_* and K, and the constants $C_j^0:=C_j^0(h_0,k_0)$, $1\le j\le 4$, in (4.16), (4.17) depend on $h_0\in\mathbb{Z}$.

Proof. Given $h_0\in\mathbb{Z}$, we multiply (4.15) by constant K in (4.13) and add it to (4.14). Then by the assumption that J_1 and J_2 are two mesh curves of type (k_0-1,k_0) and (k_0,k_0+1), we obtain

$$\begin{aligned}
F(J_2)-F(J_1)=\ &-K\sum_{h_0\in\mathbb{Z}}Q(\Gamma_{k_0,h_0})+O(1)[1+KL(J_1)]\Big\{\sum_{h_0\in\mathbb{Z}}Q(\Gamma_{k_0,h_0})\\
&+\sum_{h_0\in\mathbb{Z}}\varepsilon_*(\Gamma_{k_0,h_0})(C_1^0+\lambda_*C_2^0+C_3^0+C_4^0)s\Big\}.
\end{aligned}$$

Next, by the observation that $\sum_{h_0\in\mathbb{Z}}Q(\Gamma_{k_0,h_0})=Q(J_1)$, the equation above implies that

$$
\begin{aligned}
F(J_2)-F(J_1) &= -KQ(J_1)+O(1)(1+KL(J_1))Q(J_1)\\
&\qquad + O(1)s\sum_{h_0\in\mathbb{Z}}\varepsilon_*(\Gamma_{k_0,h_0})(C_1^0+\lambda_* C_2^0+C_3^0+C_4^0)\\
&= Q(J_1)\{K[O(1)L(J_1)-1]+O(1)\}\\
&\qquad + O(1)s\sum_{h_0\in\mathbb{Z}}\varepsilon_*(\Gamma_{k_0,h_0})(C_1^0+\lambda_* C_2^0+C_3^0+C_4^0)\\
&\le Q(J_1)\{K[O(1)M_*-1]+O(1)\}\\
&\qquad + O(1)s\sum_{h_0\in\mathbb{Z}}\varepsilon_*(\Gamma_{k_0,h_0})(C_1^0+\lambda_* C_2^0+C_3^0+C_4^0).
\end{aligned}
$$

The last inequality is an application of (4.23). We see that the term $K[O(1)M_*-1]+O(1)$ is negative, if M_* is sufficiently small and K is sufficiently large. Thus, (4.24) holds for such M_* and K. This completes the proof. □

We now establish the stability of generalized Glimm method, which is the main result of this section. We denote by $TV(\cdot)$ the total variation of a function.

Theorem 4.3 *Fix a constant state u_* and assume that the initial data $u_0=u_0(x)$ is a function of bounded variation such that*

$$||u_0-u_*||_{L^\infty} \text{ and } TV(u_0) \text{ are sufficiently small.} \tag{4.25}$$

Assume also that the mappings $A(t,x,u):=\frac{Df}{Du}(t,x,u)$ and $q(t,x,u)$ in (1.4) are smooth and such that

$$\text{the } L^1(\mathbb{R}_+\times\mathbb{R}) \text{ norm of } \frac{\partial^2 A}{\partial t\partial u},\ \lambda_*\frac{\partial^2 A}{\partial x\partial u},\ q,\ \frac{\partial q}{\partial u} \text{ are sufficiently small.} \tag{4.26}$$

Then, the approximate solutions $u_r(t,x)$ are bounded uniformly in the L^∞ and BV norms:

$$||u_r-u_*||_{L^\infty(\mathbb{R}_+\times\mathbb{R})} \le O(1)\Big(||u_0-u_*||_{L^\infty(\mathbb{R})}+TV(u_0)+C\Big), \tag{4.27}$$

$$TV(u_r(t,\cdot)) \le O(1)\Big(TV(u_0)+C\Big), \tag{4.28}$$

where

$$C := \Big|\Big|\frac{\partial^2 A}{\partial t\partial u}\Big|\Big|_{L^1(\mathbb{R}_+\times\mathbb{R})}+\lambda_*\Big|\Big|\frac{\partial^2 A}{\partial x\partial u}\Big|\Big|_{L^1(\mathbb{R}_+\times\mathbb{R})}+||q||_{L^1(\mathbb{R}_+\times\mathbb{R})}+\Big|\Big|\frac{\partial q}{\partial u}\Big|\Big|_{L^1(\mathbb{R}_+\times\mathbb{R})} \tag{4.29}$$

Furthermore, the function $u_r(t,x)$ is Lipschitz continuous in time, i.e., for $t_1,t_2>0$,

$$\int_{\mathbb{R}} |u_r(t_1,x)-u_r(t_2,x)|\,dx \le O(1)(|t_2-t_1|+s)\,(TV(u_0)+C). \tag{4.30}$$

Proof. We apply an induction argument based on Lemma 4.2 to show that the approximate solution $u_r(t,x)$ is uniformly bounded in L^∞ and total variation. First, we show that the condition (4.23) in Lemma 4.2 holds under the assumptions (4.25), (4.26). By induction, given $k_0 \in \mathbb{N}$, we let $J_{k_0-1/2}$ denote the mesh curve of type (k_0-1, k_0). For $k_0 = 1$, we see that

$$F(J_{1/2}) \le O(1)\big(TV(u_0) + K[TV(u_0)]^2\big). \tag{4.31}$$

This means that there exists a positive constant M_*, as described in (4.23), such that $F(J_{1/2}) \le M_*$, and in particular, $L(J_{1/2}) \le M_*$ if $TV(u_0)$ is sufficiently small. Next, suppose that

$$L(J_{k+1/2}) \le M_* \quad \text{for} \;\; k = 0, 1, \dots, k_0 - 1. \tag{4.32}$$

We intend to show that (4.32) still holds for $k = k_0$. Since $J_{k_0-1/2}$ is a mesh curve of type (k_0-1, k_0), this implies that $J_{k_0+1/2}$ is a mesh curve of type (k_0, k_0+1) so that Lemma 4.2 can be applied. Therefore we obtain

$$\begin{aligned} F(J_{k_0+1/2}) &\le F(J_{k_0-1/2}) + O(1)s \sum_{h_0\in\mathbb{Z}} \varepsilon_*(\Gamma_{k_0,h_0})(C_1^0 + \lambda_* C_2^0 + C_3^0 + C_4^0) \\ &\;\;\vdots \\ &\le F(J_{1/2}) + O(1)s \sum_{k=1}^{k_0} \sum_{h_0\in\mathbb{Z}} \varepsilon_*(\Gamma_{k,h_0})(C_1^0 + \lambda_* C_2^0 + C_3^0 + C_4^0). \end{aligned}$$

Then, by

$$\sum_{h_0\in\mathbb{Z}} \varepsilon_*(\Gamma_{k,h_0}) = L(J_{k-1/2}), \qquad k \in \mathbb{N},$$

this leads to

$$F(J_{k_0+1/2}) \le F(J_{1/2}) + O(1) \sum_{k=1}^{k_0} sL(J_{k-1/2}) \sup_{h_0\in\mathbb{Z}} (C_1^0 + \lambda_* C_2^0 + C_3^0 + C_4^0). \tag{4.33}$$

Next, by (4.31)-(4.33) we find

$$\begin{aligned} F(J_{k_0+1/2}) \;&\le O(1)(1 + K\, TV(u_0))\, TV(u_0) \\ &\qquad + O(1) M_* \sum_{k=1}^{k_0} \sup_{h_0\in\mathbb{Z}} (C_1^0 + \lambda_* C_2^0 + C_3^0 + C_4^0)s. \end{aligned} \tag{4.34}$$

From the definitions of C_j^0 and the assumption that the constant C in (4.29) is finite, we see that

$$\lim_{r\to 0} \sum_{k=1}^{\infty} \sup_{h_0\in\mathbb{Z}} (C_1^0 + \lambda_* C_2^0 + C_3^0 + C_4^0)s = C. \tag{4.35}$$

Therefore, from (4.34) and (4.35) we obtain the inequality

$$F(J_{k_0+1/2}) \leq O(1)\{(1 + K\,TV(u_0))\,TV(u_0) + M_*C\}, \tag{4.36}$$

and in particular,

$$L(J_{k_0+1/2}) \leq O(1)\{(1 + K\,TV(u_0))\,TV(u_0) + M_*C\}. \tag{4.37}$$

We note that the functional L in (4.37) only depends on the constants M_*, C and the total variation of u_0, thus it enables us to choose $TV(u_0)$ and C sufficiently small such that $O(1)(1 + K\,TV(u_0))\,TV(u_0) \leq \frac{M_*}{2}$ and $O(1)\,CM_* \leq \frac{M_*}{2}$ and this implies that

$$L(J_{k_0+1/2}) \leq M_*.$$

Therefore (4.32) holds for $k = k_0$, we just showed that $L(k_0 + 1/2)$ has uniform bound for all $k_0 \in \mathbb{N}$, which implies that functional L of $u_r(t, x)$ has global bound. Since L is a functional equivalent to the total variation of $u_r(t, x)$, we prove that the total variation of $u_r(t, x)$ has an uniform bound for all $t \geq 0$ and all finite $r > 0$, so as well the L^∞ norm of $u_r(t, x)$. To prove (4.28), we apply (4.4), (4.5) to $u_r(t, x)$ and we use the fact that $TV(u_r(k_0s, \cdot)) = O(1)\,F(J_{k_0+1/2})$ to (4.36), then (4.28) is established. For the proof of (4.27) and (4.30), we follow the lines of proof in [10]. The proof is completed. □

We note that if $\frac{\partial^2 A}{\partial t \partial u}$, $\lambda_* \frac{\partial^2 A}{\partial x \partial u}$, q and $\frac{\partial q}{\partial u}$ in (4.26) belong to L^∞, then inequalities (4.27), (4.28) and (4.30) remain valid in a finite interval $[0, T]$ with T sufficiently small.

5. Convergence of the generalized Glimm method

In Section 4 we established the BV stability of the scheme together with a time continuity property. By Helly's theorem ,there exists a subsequence of approximate solutions, still denoted by $\{u_r(t, x)\}$ and converging strongly in L^1_{loc} to a limit function $u = u(t, x)$. Moreover, by the estimates (4.23), (4.24), the function u is uniformly bounded and is of bounded variation in x. We now prove that the limit u is indeed an entropy solution of the Cauchy problem. The proof relies on the error estimate derived in Section 2.

Theorem 5.1 *Suppose that the initial data $u_0(x)$ is sufficiently close to a constant state in L^∞ and BV, and that the L^1 norms of $\frac{\partial^2 A}{\partial x \partial u}$, $\frac{\partial^2 A}{\partial t \partial x}$, q, and $\frac{\partial q}{\partial u}$ are sufficiently small in $\mathbb{R}_+ \times \mathbb{R}$. Let $\{u_r(t, x) : r > 0\}$ be the sequence of approximate solutions constructed by the generalized Glimm scheme (4.3)-(4.5). Then, for any equidistributed sequence $\{a_k\}_{k \in \mathbb{N}}$, there exists a subsequence of $\{u_r(t, x)\}$ converging in L^1_{loc} to a function $u = u(t, x)$ which is an entropy solution of the Cauchy problem (1.1), (1.2).*

Remark 5.2. Assume that $\mathcal{U}$ is a convex subset of R^p, we say that (U, F), $U : \mathcal{U} \in \mathbb{R}^p \to \mathbb{R}$ and $F : \mathbb{R}_+ \times \mathbb{R} \times \mathcal{U} \to \mathbb{R}$, is an entropy pair of the system (1.1) if U is a convex function on $\mathcal{U}$ and

$$\frac{\partial F}{\partial u} = \frac{DU}{Du} \frac{\partial f}{\partial u} \quad \text{on } \mathbb{R}_+ \times \mathbb{R} \times \mathcal{U}.$$

Furthermore, a function $u : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}^p$ is called an entropy solution of (1.1) if $u = u(t, x)$ is a weak solution of (1.1) satisfying

$$\partial_t U(u) + \partial_x (F(t, x, u)) \leq \frac{DU}{Du}(u)\{g(t, x, u) - (\partial_x f)(t, x, u)\} + (\partial_x F)(t, x, u) \tag{5.1}$$

in the sense of distributions, for every entropy pair (U, F).

Proof. The proof is based on the result of Proposition 2.1. Let $\{u_r(t, x)\}$ denote a sequence of approximate solutions constructed by generalized Glimm scheme (4.3)-(4.5). Then, by the stability result and Helly's theorem, there exists a subsequence of $\{u_r(t, x)\}$ converging almost everywhere to a function $u \in L^1_{loc}$ with bounded total variation. Given any test-function $\theta : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ with we define the residual of u_r as

$$\mathcal{R}(u_r, \theta) := \int_{\mathbb{R}_+} \int_{\mathbb{R}} \{u_r \partial_t \theta + f(t, x, u_r)\partial_x \theta + g(t, x, u_r)\theta\} dx \; dt + \int_{-\infty}^{\infty} u_0(x)\theta(0, x) dx$$

Note that u is a weak solution to (1.1), (1.2) if and only if $\mathcal{R}(u, \theta) = 0$ for any test-function θ. By Lebesgue's theorem, we see that

$$|\mathcal{R}(u_r, \theta) - \mathcal{R}(u, \theta)| \to 0 \quad \text{as } r \to 0.$$

Thus, to show that u is a weak solution of (1.1), (1.2), it is equivalent to show that $\mathcal{R}(u_r(t, x), \theta)$ tends to zero as r vanishes. To show this, we first let $\chi^{k_0, h_0}_{supp\ (\theta)}$ denote a characteristic function having the same support as the test-function θ.

Then, by construction of $u_r(t,x)$ and by (2.8), we can write

$$
\begin{aligned}
&\mathcal{R}(u_r,\theta)\\
&=\sum_{k_0=0}^{\infty}\sum_{\substack{h_0+k_0\\ \text{even}}}\int_{k_0 s}^{(k_0+1)s}\int_{(h_0-1)r}^{(h_0+1)r}\Big(u_r\partial_t\theta+f(t,x,u_r)\partial x\theta+g(t,x,u_r)\theta\Big)\,dx\,dt\\
&=\sum_{k_0=0}^{\infty}\sum_{\substack{h_0+k_0\\ \text{even}}}O(1)(s^2+r^2)(s+r+|u_{k_0,h_0+1}-u_{k_0,h_0-1}|)\chi^{k_0,h_0}_{supp\ (\theta)}\\
&\quad+\Bigg(\sum_{k_0=0}^{\infty}\sum_{\substack{h_0+k_0\\ \text{even}}}\bigg(\int_{(h_0-1)r}^{(h_0+1)r}u_r((k_0+1)s-,x)\theta((k_0+1)s,x)dx\\
&\qquad-\int_{(h_0-1)r}^{(h_0+1)r}u_r(k_0s+,x)\theta(k_0s,x)dx\bigg)+\int_{-\infty}^{\infty}u_0(x)\theta(0,x)dx\Bigg)\\
&\quad+\sum_{k_0=0}^{\infty}\sum_{\substack{h_0+k_0\\ \text{even}}}\bigg(\int_{k_0s}^{(k_0+1)s}f(t,(h_0+1)r,u_r(t,(h_0+1)r-))\theta(t,(h_0+1)r)dt\\
&\qquad-\int_{k_0s}^{(k_0+1)s}f(t,(h_0-1)r,u_r(t,(h_0-1)r+))\theta(t,(h_0-1)r)dt\bigg).
\end{aligned}
\tag{5.2}
$$

Let $\Omega_1(r)$, $\Omega_2(r)$ and $\Omega_3(r)$ denote the terms on the right hand side of (5.2), respectively. We first estimate $\Omega_1(r)$. By a direct calculation and (4.2), (4.28), we obtain

$$
\begin{aligned}
\Omega_1(r)=&O(1)r+\sum_{k_0=0}^{\infty}\sum_{\substack{h_0+k_0\\ \text{even}}}O(1)(s^2+r^2)(|u_{k_0,h_0+1}-u_{k_0,h_0-1}|)\chi^{k_0,h_0}_{supp\ (\theta)}\\
\le&O(1)r+\sum_{k_0\in\mathbb{N}}O(1)(s^2+r^2)(T.V.\{u_0(x)\}+C)\chi^{k_0,h_0}_{supp\ (\theta)}\\
\le&O(1)r.
\end{aligned}
\tag{5.3}
$$

Next we calculate $\Omega_3(r)$. By the property of the Lipschitz continuity of f, q and (2.4), (4.5), we obtain

$$
\begin{aligned}
&\Omega_3(r)\\
&=O(1)\sum_{k_0=0}^{\infty}\sum_{\substack{h_0+k_0\\ \text{even}}}\int_{k_0s}^{(k_0+1)s}|u_r(t,(h_0+1)r+)-u_r(t,(h_0-1)r-)|\cdot(\chi^{k_0,h_0}_{supp\ (\theta)})dt\\
&=O(1)\sum_{k_0=0}^{\infty}\sum_{\substack{h_0+k_0\\ \text{even}}}\bigg(\int_{k_0s}^{(k_0+1)s}t|q(k_0s,(h_0+2)r,\tilde{u}_{k_0,h_0+1})-q(k_0s,h_0r,\tilde{u}_{k_0,h_0+1})|\cdot(\chi^{k_0,h_0}_{supp\ (\theta)})dt\bigg)\\
&=O(1)\sum_{k_0}\sum_{h_0}\int_{k_0s}^{(k_0+1)s}t\ r\ \cdot(\chi^{k_0,h_0}_{supp\ (\theta)})\ dt.
\end{aligned}
$$

It follows that

$$\Omega_3(r) = O(1)r. \tag{5.4}$$

It remains to estimate $\Omega_2(r)$. It is a standard matter to check that

$$\begin{aligned}\Omega_2(r) = &-\sum_{k_0=1}^{\infty}\sum_{\substack{h_0+k_0\\ \text{even}}}\int_{(h_0-1)r}^{(h_0+1)r}[u_r](k_0 s,x)\theta(k_0 s,x)dx\\ &-\int_{-\infty}^{+\infty}(u_r(0,x)-u_0(x))\theta(0,x)dx\end{aligned}$$

where $[u_r](k_0 s,x) := u_r(k_0 s+,x) - u_r(k_0 s-,x)$. We let $J(\{a_k\},r,\theta)$ denote the term

$$\sum_{k_0=1}^{\infty}\sum_{\substack{h_0+k_0\\ \text{even}}}\int_{(h_0-1)r}^{(h_0+1)r}[u_r](k_0 s,x)\theta(k_0 s,x)dx.$$

By the construction of $u_r(t,x)$ in (4.3), we see that the term $\int_{-\infty}^{+\infty}(u_r(0,x) - u_0(x))\theta(0,x)dx$ on the right hand side of (5.5) vanishes as r tends to zero. In addition, by a result of Liu [19] we obtain that, for any equidistributed sequence $\{a_k\}_{k\in\mathbb{N}}$, $J(\{a_k\},r,\theta)$ tends to zero as r approaches to zero. This implies that

$$\Omega_2(r) \to 0 \quad \text{as } r \to 0 \tag{5.5}$$

for every equidistributed sequence $\{a_k\}_{k\in\mathbb{N}}$. We refer the reader to [19] for the details of the estimate of $\Omega_2(r)$. Finally, by (5.3), (5.4) and (5.5), we obtain

$$\mathcal{R}(u_r,\theta) \to 0 \text{ in } L^1 \quad \text{as } r \to 0,$$

which means that the limit function u satisfies $\mathcal{R}(u,\theta) = 0$. Therefore, u is a weak solution of the Cauchy problem (1.1), (1.2).

To prove that u is an entropy solution satisfying the entropy inequality (5.1), it is equivalent to show that, for any entropy pair (U,F) and test-function $\theta \geq 0$, the function u satisfies

$$\int_{\mathbb{R}_+}\int_{\mathbb{R}} U(u)\theta_t + F(t,x,u)\theta_x + P(t,x,u)\theta dxdt + \int_{\mathbb{R}} U(u_0(x))\,\theta(x,0)\,dx \geq 0, \tag{5.6}$$

with

$$P(t,x,u) := \frac{DU}{Du}\cdot(g - \frac{\partial f}{\partial x})(t,x,u) + (\partial_x F)(t,x,u).$$

We note that the result of Proposition 2.1 can be applied to show that $u(t,x)$ satisfies (5.6) for any entropy pair (U,F). In turn, this implies that u is an entropy solution of the Cauchy problem (1.1), (1.2), and the proof of Theorem 5.1 is completed.

□

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